Freezing mechanisms for percolation on trees

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FREEZING MECHANISMS FOR PERCOLATION ON TREES

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«El tiempo se bifurca perpetuamente
hacia innumerables futuros...»

Jorge Luis Borges
– El laberinto de caminos que se bifurcan.

A mis padres.

Abstract

This thesis investigates various aspects of frozen percolation on trees. The frozen percolation model on the 3-regular tree was introduced by Aldous in [6]. This process can be heuristically described briefly as follows: consider bond Bernoulli percolation on the 3-regular tree where edges open one after the other. As soon as a cluster gets to have an infinite number of edges it freezes, which means that its neighboring edges are not allowed to open anymore. Aldous proved that a model with the previous mechanism can be rigorously defined. A modification of this model of Aldous defined on the sites of the 3-regular tree was introduced and proved to exist by Brouwer in [19].

In the first part of the thesis we review the ideas of Aldous to construct the frozen percolation process on the 3-regular tree and we discuss how to extend them to the more general setting of Galton-Watson trees. In a second part, we study the geometry of clusters in the frozen percolation model on Galton-Watson trees as well as in the modified version of Brouwer. In this part we discuss the invariance and non-invariance of the law of non-frozen clusters with respect to time.

In the last chapters of the thesis we focus on the convergence of the finite parameter versions of the frozen percolation models on d-regular trees. In the finite parameter models of frozen percolation, clusters get frozen as soon as they reach size $N \in \mathbb{N}$. The model of frozen percolation on edges of the binary tree was introduced by van den Berg, Kiss and Nolin in [12] and it was proved to converge to Aldous' model as $N \to \infty$. We present a different proof of this

result, and in the final chapter, we use our approach to show the $N \to \infty$ convergence also for the modified frozen percolation on the binary tree.

Zusammenfassung

Die vorliegende Dissertation befasst sich mit verschiedenen Aspekten der gefrorenen Perkolation auf Bäumen. Das Modell der gefrorenen Perkolation auf 3-regulären Bäumen wurde von Aldous in [6] eingeführt. Dieser Prozess kann heuristisch wie folgt beschrieben werden: man betrachte Ber-noulli-Perkolation auf den Kanten eines 3-regulären Baumes, wobei die Kanten sich eine nach der anderen öffnen. Sobald ein offenes Cluster unendlich viele Kanten besitzt, friert es. Das bedeutet, dass seine benachbarten Kanten danach nicht mehr geöffnet werden können. Aldous bewies, dass ein Modell mit diesen Eigenschaften existiert und rigoros definiert werden kann. Ein ähnliches Modell wie das von Aldous, das stattdessen auf den Knoten des 3-regulären Baumes definiert ist, wurde von Brouwer in [19] eingeführt und rigoros konstruiert.

Im ersten Teil dieser Arbeit blicken wir auf die Ideen zurück, die von Aldous verwendet wurden, um den Prozess der gefrorenen Perkolation auf dem 3-regulären Baum zu konstruieren, und diskutieren, wie man sie zum allgemeineren Fall von Galton-Watson Bäumen erweitern kann. In einem zweiten Teil untersuchen wir dann die Geometrie der Cluster für die gefrorene Perkolation auf Galton-Watson Bäumen und auch für die modifizierte Version von Brouwer. In diesem Teil diskutieren wir die Invarianz und Nichtinvarianz der Verteilung der ungefrorenen Cluster bezüglich der Zeit.

In den letzten Kapiteln dieser Arbeit konzentrieren wir uns auf die Konvergenzeigenschaften der Versionen des gefrorenen Perkolationsmodells mit endlichen Parametern auf d-regulären Bäumen.

Im Fall endlicher Parameter frieren die Cluster, sobald sie Grösse $N \in \mathbb{N}$ erreichen. Das entsprechende Modell der gefrorenen Perkolation auf den Kanten des
binären Baumes wurde von van den Berg, Kiss und Nolin in [12] eingeführt. Darin wurde auch gezeigt, dass dieses Modell für $N \to \infty$ gegen Aldous' Modell konvergiert. Wir geben einen anderen Beweis dieses Resultats und im letzten Kapitel verwenden wir unsere Herangehensweise, um die Konvergenz für $N \to \infty$ auch für das modifizierte Modell der gefrorenen Perkolation zu zeigen.

Contents

Chapter 0

Introduction

Generalities about polymerization and gelation

The term *polymer* refers to a molecule composed of multiple repeating units named *monomers*. The chemical process by which monomers undergo a reaction to form a polymer is called *polymerization*.

A deterministic kinetic theory of polymerization was proposed by Smo-
luchowski in 1918, [53]. Smoluchowski's model is defined in terms of *concentration functions* $c(k,t)$, $k \in \mathbb{N}$ and $t \in [0,\infty)$, where $c(k,t)$ stands for the number of polymers formed by k units per unit of volume at time t. The polymerization model proposed by Smoluchowski goes as follows: considering a polymer of size i (consisting of i monomers), the rate at which this polymer coalesces with a polymer of size j (thereby forming a polymer of size $i+j$) is proportional to the concentration $c(j,t)$ of polymers of size j at time t, where the constant of proportionality is given by a symmetric function $K(i,j)$ often referred to as the coagulation kernel. The rate at which polymers of size i and j coalesce is then equal to

$$\frac{1}{2}c(i,t)c(j,t)K(i,j). \tag{0.1}$$

The first natural kernel that one can probably think of in this setting is the multiplicative kernel $K(i,j) = ij$. Indeed, ij can represent the number of possible links between a polymer of size i and a polymer of size j. From the point of view of statistical mechanics the polymerization model under this kernel turns to be interesting because of the existence of a phase transition, which, in the parlance of polymerization theory, is referred to as *gelation*, and which we now discuss.

Assume for simplicity that, by suitable normalization, the initial concentration of monomers $\sum_{i=0}^{\infty} ic(i,0)$ is equal to 1. Then for each time t, the quantity $ic(i,t)$ can be interpreted as the probability that a randomly chosen monomer in the solution at time t does belong to a polymer of size i. In particular, the total concentration of monomers that belong to a *finite* polymer at time t is equal to the quantity

$$1 - \theta(t) := \sum_{i=1}^{\infty} ic(i,t). \tag{0.2}$$

The phase transition can then be described as follows (for references see for example [20], [22], [31], [42], [43], [54]): There exists a special time usually denoted by T_{gel} such that $\theta(t) = 0$ for all $t < T_{gel}$. In other words, all monomers do still belong to finite polymers. However, when $t > T_{gel}$, the function $\theta(t)$ becomes positive, and it increases with time. This corresponds to the formation of infinite mass polymers referred to as *the gel*.

Up to time T_{gel} the model described through the coalescence rates in (0.1) is governed by the so-called Smoluchowski equation, for $k \geq 1$

$$\frac{d}{dt}c(k,t) = \frac{1}{2} \sum_{\substack{i,j \geq 1: \\ i+j=k}} c(i,t)c(j,t)ij - c(k,t) \sum_{i=1}^{\infty} c(i,t)ik. \tag{0.3}$$

However, when $t > T_{gel}$, the previously described rules are not sufficient to describe the evolution. Indeed, one has to decide whether the finite polymers are allowed to interact and coalesce with the gel or not.

- The first case to consider is that finite polymers do coalesce with the gel, and the natural rate at which a polymer of size k coalesces with the gel is then $k\theta(t)$. This led Flory [25, 26, 27, 28] to consider the model governed by the equation,
(0.4)
$$\frac{d}{dt}c(k,t) = \frac{1}{2}\sum_{\substack{i,j\geq 1:\\ i+j=k}} c(i,t)c(j,t)ij - c(k,t)\sum_{i=1}^{\infty} c(i,t)ik - k\theta(t),$$
where the interaction between gel and finite polymers is taken into account in the last term in (0.4).

- Alternatively, one can decide that a finite polymer is only allowed to coalesce with other finite polymers. In other words, the infinite polymers that belong to the gel can be thought of as *frozen*. In that case, Stockmayer [52] argued that the evolution of the concentrations of polymers is governed by Smoluchowski's equation (0.3) also after T_{gel}.

While the evolutions described by these equations are totally deterministic, they should appear in some way as the large scale limit of some random microscopic model of coalescing polymers, which we now discuss.

Multiplicative coalescent, percolation on the complete graph and gelation

Let us first briefly recall how one can view the multiplicative coalescence kernel via percolation on a large complete graph: Consider the complete graph with N vertices and consider independent identically distributed exponential random clocks with parameter $1/N$ assigned to each edge in the graph. When the clock of an edge rings, we say that this edge becomes activated. At time t the graph configuration generated by active edges is just percolation on this complete graph, i.e., the Erdös-Rényi random graph with parameters N and $p_N := 1 - \exp(-t/N)$. Denote by $c_N(k,t)$ the number of activated connected components of size k at time t divided by N. The model defined in this way is described by the rates in (0.1), so we might expect that the concentration functions $c_N(k,t)$ converge to the solutions of Smoluchowski's equation before the gelation time. After the gelation time, they turn out to converge to the solution of Flory's equation (see [3] and [30]) which mirrors the fact that polymers are still allowed to coalesce with the so-called giant cluster in the supercritical Erdös-Rényi model.

David Aldous in [2] proposed the following modification of the previous model on the complete graph in order to approximate the case where polymers do not interact with the gel: consider again the complete graph with N vertices and i.i.d. exponential clocks with parameter $1/N$ assigned to each edge in the graph. When the clock between a pair of vertices rings the link between them gets activated if and only if both vertices are in an active connected component with less than $\alpha(N)$ vertices. Active connected components with a number of vertices exceeding the threshold function $\alpha(N)$ are thought to be frozen, they correspond with the inert gel in Stockmayer's model. Aldous conjectured that the process of concentrations analogous to the ones defined for the random graph model converge to the solutions of Smoluchowski's equation also after the gelation time. This conjecture was already proved in [32] assuming $1 \ll \alpha(N) \ll N$. Further information about the shape of typical small clusters in the supercritical phase were then obtained in [46]. Recall that $N^{2/3}$ is the order of a typical size of a critical cluster in the Erdös-Rényi graph, for this and more results in the Erdös-Rényi graph, and the emergence of the giant cluster, see for instance [21] or [37], for another closed related model converging to the solution of Smoluchowski equation as well see [50]. When $N^{2/3}\log^3 N \ll \alpha(N) \ll N$, the geometric information about the shape of the polymers (this is part of Theorem 3 in [46] Merle & Normand) is as follows: At time $t \geq 0$, pick uniformly at random a particle i in the solution (i.e., conditioned not to be in the gel), and let $\mathcal{C}_{typ}(t)$ denote the graph rooted at i consisting of the active connected component of i. Then, for any rooted finite tree T, the probability that $\mathcal{C}_{typ}(t) = T$ tends as $N \to \infty$ to the probability that a Galton-Watson tree with Poisson offspring distribution with mean

$\min(t, 1)$ is equal to T. The result for $t < 1$ is not surprising as it describes the law of subcritical percolation clusters in the Erdös-Renyi model. The result for $t > 1$ is a little more surprising: It says that (in the $N \to \infty$ limit), the law of a typical cluster in the solution does not depend on $t \geq 1$, and is exactly the same as at the critical time $t = 1$. This feature will be a recurrent theme in the present thesis.

Relation to percolation on d-regular trees, Aldous' frozen percolation model

The reader might have already noticed that the choice of the multiplicative kernel $K(i, j) = ij$ does not seem to describe exactly the actual polymerization in chemistry. This is illustrated here by the fact that in this natural representation of the coalescent kernel $K(i, j) = ij$ via large complete graphs, the "polymers" that one constructs are not so realistic in view of chemical interpretations, because the degree of a given site is not bounded (whereas one learns in school that the degree of a Carbon atom in the graph representing an alkane molecule C_iH_{2i+2} is 4, so that this degree in the subtree consisting only of the Carbon molecules is at most 4). In that alkane polymerization setting, it is therefore more natural to consider the case where an alkane with i Carbon atoms will coalesce with an alkane with j Carbon atoms and form an alkane with $i + j$ Carbon atoms with a rate given by $K(i, j) = (2i + 2)(2j + 2)$ that corresponds to the number of possible available pairs of $C - H$ connections. Of course here, one still discards the "steric/geometric" aspects of the problem (i.e., the fact that a polymer chain has to be self-avoiding in space).

More generally, if one consider chains of k-valent atoms instead of tetra-valent ones, the natural kernel to consider would be

$$K_k(i, j) = (i(k-2) + 2)(j(k-2) + 2).$$

The simplest non-trivial case is here $k = 3$, which corresponds to trivalent atoms, where $K_3(i, j) = (i + 2)(j + 2)$. The case $K(i, j) = ij$ can then be viewed as the limit as $k \to \infty$ of the kernel K_k/k^2, which is a time-rescaled version of K_k.

Maybe motivated by the fact that this could provide a nice interpretation for the coalescence kernel K_3, Aldous proposed in [6] a random percolation process on the binary tree that enables to geometrically represent this particular coalescent kernel. This model, which will be closer to the main focus of the present thesis is defined in the infinite binary tree $\mathbb{T} = (V, E)$, that is an infinite connected graph with no cycles where all vertices have degree three. Consider a collection of independent identically distributed random clocks distributed uniformly on $[0, 1]$ assigned to each edge of the binary tree. When a clock rings, this edge tries to get activated. There are two natural rules that one can choose in order to decide if this attempt is a success:

(i) One decides that all edges that try to get activated actually do get activated. This model is obviously just the natural increasing coupling of all ordinary percolation processes with parameter t on the tree. One can alternatively represent this process as a realization of the coalescent kernel $K_3(i, j) = (i + 2)(j + 2)$. Indeed, the collection of finite clusters that one discovers at time t, when moving along a branch say, will be a collection of independent identically distributed finite trees. In some sense, if one considers the representation of the coalescent kernel K_3 via an infinite collection of coalescent individuals (see e.g. [4], [5], [15], [23], [29], [40], [49] and references therein for background), the tree picture does represent this collection of independent trees assembled in the way in which they will actually "eventually" merge. Note that in this model, one authorizes the fact that an edge can be opened, even when one of its endpoints is already in an infinite cluster. This corresponds to the fact that one is actually looking at a version of Flory's model (so one would loosely speaking need to add a coalescent kernel with the infinite clusters to K_3 in order to represent the percolation model in the supercritical regime).

(ii) Aldous introduced a model more in the spirit of Stockmayer's model, where the gel is not allowed to interact with the solution: At the time at which the clock corresponding to an

> edge rings, this edge gets activated if and only if its both end-vertices belong to finite active connected components. In other words, if when the clock rings, at least one of the end-vertices of the edge does already belong to an infinite active cluster, then the edge stays inactive (and remains inactive forever – this is an important non-trivial feature here!). Intuitively, infinite open connected components in the tree get instantaneously frozen as soon as they appear, and they freeze all their boundary edges, i.e., they get surrounded by edges which will always remain inactive forever. Of course, this phenomenon of "instantaneous freezing" can appear at first somewhat mysterious, but as we shall see when we will review Aldous' arguments, the phenomenon in the special case of regular trees is actually not so difficult to understand. Mind that (we will recall the simple argument by Benjamini and Schramm) that a frozen percolation process of the type that Aldous constructs in the infinite tree can not exist in the infinite square lattice.

Note that in both models, the gelation time T_{gel} would correspond to the critical probability t_c for percolation on the tree (which is $1/2$ for the 3-regular tree). In view of the solution-gel interpretation and motivation, one can think of Aldous' model as follows: One has a solution with only monomers (with exactly one trivalent atom in each of them) at time 0. These monomers will coalesce into polymers via the kernel K_3, so that a molecules with i monomers will coalesce with molecules with j monomers at a rate $(i+2)(j+2)$. However, after the gelation time, infinite chains will appear, and the following rule is applied: If a finite polymer in the solution attempts to connect to the gel via one of its available outgoing possible edges, then this attempt fails *and this outgoing edge can not be used anymore in the future in order to create a connection*. It is worth stressing that this means that in our chemical interpretation, the gel does in fact interact with the solution: The outgoing connection that tried to connect to the gel and failed to get activated (because of the rule that a finite polymer cannot coalesce with the gel) can not be used anymore. So, in Aldous' model, the gel (once created) is somehow preventing some further possible coalescences between polymers in the solution by disactivating some possible connection points. (If one would measure the “size” of a polymer by the number of its available (not yet destroyed) possible outgoing connections, then one could view the interaction with the gel as an “erosion” term where when a polymer of size i interacts with the solution, its size decreases by 1. However, this would not enable to keep track of the actual “real” polymers that are left in the solution.) We can note that after the gelation time, a positive fraction of the polymers will “burn down” all their possible outgoing connections before coalescing with any other polymer, and will remain forever finite. So, at time 1, a positive fraction of the initial monomers will not belong to the gel. Aldous shows that the probability that a monomer does not belong to the gel at time t is $1/2t$ when $t > 1/2$ (and 1 when $t \leq 1/2$). At time 1, this probability is $1/2$.

In [6], Aldous does also provide some geometric information about the clusters in this frozen percolation model. Of course, up to the critical time $1/2$, things are very easy and the model obviously coincides with Bernoulli percolation with increasing parameter on the binary tree, so that the finite clusters are subcritical. However, Aldous points out that for this frozen percolation model, for any time $t > 1/2$ the law of a finite active cluster is that of the law of a critical cluster in Bernoulli percolation. In other words, the law of a finite active cluster at time $t \geq 1/2$ does not depend on t (this fact can be related with the aforementioned result of Merle and Normand). This again contrasts with ordinary Bernoulli percolation, where a typical finite cluster in the supercritical regime tends to be much smaller than at the critical point. In other words, the parameter t needs to be "fine tuned" to exhibit a critical behaviour in ordinary percolation, while frozen percolation on the binary tree finite clusters exhibits critical behavior at all times $t \geq 1/2$. Aldous also shows that when $t > 1/2$, individual frozen clusters are distributed like the incipient infinite critical percolation cluster.

Aldous also states in the final section of his paper [6] that his results and proofs can be generalized to the case of d-ary trees. As one can also expect from our previous remark about the limit of

K_k/k^2 as $k \to \infty$, the appropriately taken limit as $d \to \infty$ of the frozen percolation model on d-ary trees will give rise to the model on the complete graph that we mentioned above.

This frozen percolation process on trees, and some of its variations that we will describe below will be the main topic of the present thesis. In the following three paragraphs, we will present some modifications of the model, and somewhat informally describe some of the results that we will be deriving.

Generalization to Galton-Watson trees

In view of Aldous' construction and of the original motivation to understand certain solutions with creation of gel, it appears natural to see whether Aldous' construction can be generalized to the case of Galton-Watson trees. The fact that Aldous' construction is based on recursive ideas, where one discovers the tree and the percolation progressively seems to indicate that the main ideas of his proof can be made to work also in this case. We shall see that indeed:

Proposition 1.

- *The frozen percolation process on a supercritical Galton-Watson tree exists, and one can compute the "freezing" density as a function of t (and of the law of the Galton-Watson tree offspring distribution), using arguments that are analogous to the case of regular trees.*
- *Furthermore, the law of the tree in the solution at a given time t is that of some Galton-Watson tree conditioned to be finite, where the law of that Galton-Watson tree offspring distribution does seem to depend on t in the general case (so this differs from the case of d-regular trees).*
- *For all $t > t_c$, the clusters in the solution are distributed according to the law of a* critical *Galton-Watson tree (critical in the sense that the expected number of offsprings is equal to* 1*). As we shall see, this implies that in the special case of geometric offspring distributions, the law of these finite clusters in the solution for $t > t_c$ does in fact not depend on t.*

Brouwer's frozen site-percolation model on the binary tree.

In the previously mentioned model, freezing always occurred on edges. However, one could argue that the exact coalescence kernel that are considered here do not represent exactly/faithfully the actual mechanism at work (one can think of catalytic effects say – one might imagine that one needs some small monomers in order to ignite the reaction, of course these interpretations are here to stimulate the mathematician's imagination, and we do not claim here that they represent some concrete chemical reality!). Anyway, a natural variant introduced by Rachel Brouwer in [19] is when the percolation (and the freezing) occurs on the sites of the binary tree rather than on its edges. Adapting the ideas of Aldous, she was able to construct the infinite volume process that can be loosely speaking described as follows: Let $\{\tau_v\}_{v\in V}$ be an independent identically distributed family of random variables uniformly distributed on $[0,1]$, indexed by the set of vertices of the tree. At any time t a vertex can be in one of three possible states, white (inactive), green (active), or red (frozen). Until the clock of a vertex rings, it is inactive (white). When the clock rings the vertex gets activated and turns green. However, as soon as a green cluster becomes infinite, it instantaneously switches to red. Here we see that (as opposed to Aldous' model), when $t = 1$, no vertex is still white (in Aldous' model, some edges were prevented from opening). Brouwer showed that the probability that a vertex is red at a time $t \in [0,1]$ is again simple function of time. When $t < 1/2$ it is zero, and when $t \in [1/2, 1]$, it is $\log(2t)$. In particular, the proportion of red sites at time 1 is $\log 2$.

Just as in the Galton-Watson case, it is no longer true that the asymptotic law of a green cluster at a supercritical time is exactly that of a critical cluster in Bernoulli site percolation (see Lemma 5.1 in [19]). However, we will point the following stationarity:

Proposition 2. *When one instead considers the subtree consisting of the "inner vertices" (with two offsprings) of the actual tree in the solution – then the law of that subtree is stationary.*

This result is in the spirit of aspects of our aforementioned result on Galton-Watson trees (the fact that the law of the trees in the solution are always "critical").

Approximations of Aldous' model via cut-offs

The construction of the frozen percolation process on the tree is very elegant, but it can appear somewhat mysterious. One hands-on way to try to understand it better and to test whether it can be relevant in the study of large finite models is to try to approximate via a more "finite-range" version which does not involve infinite clusters. One can modify the model so that instead of freezing infinite connected components, one freezes clusters as soon as their size exceeds a certain threshold N (we will discuss what notions of "size" are possible here). There is no definition problem for this process. Its existence is automatically provided by the classical theory of interacting particles systems (basically because the process is a finite range interacting particle system; for a reference of such existence results see [44]). This finite frozen percolation on the binary tree has been introduced by van den Berg, Kiss and Nolin in [12], where the authors prove a weak convergence of this model as $N \to \infty$ to Aldous' model. The result is actually proved for a certain class of good size functions that includes for instance the diameter of the volume of the considered tree.

The convergence proved in [12] is stated for the planted binary tree, that is the infinite graph with no cycles with all vertices having degree three, except for one vertex having degree one. Let us first put down some notation before stating their result. The edge containing this vertex is called the root edge and we denote it as e_0. We denote the planted binary tree as $\overline{\mathbb{T}}$ and by $\overline{\mathcal{T}}_0$ the set of connected subgraphs of $\overline{\mathbb{T}}$ containing the root edge e_0 . Define $\mathcal{C}_t$ to be the active connected component of e_0 at time t in the frozen percolation process in $\overline{\mathbb{T}}$. Denote by $\mathbb{P}_N^{(s)}$ the law of the frozen percolation process with good size function s in $\overline{\mathbb{T}}$ and by $\mathbb{P}_\infty$ the law of Aldous' frozen percolation process in $\overline{\mathbb{T}}$. Then, van den Berg, Kiss & Nolin (this is part of Theorem 2 in [12]) showed that when s be a good size function for the planted binary tree, then, for all $t \in [0,1]$,

$$\mathbb{P}_N^{(s)}\left(\mathcal{C}_t = \mathbf{T}\right) \longrightarrow \mathbb{P}_\infty\left(\mathcal{C}_T = \mathbf{T}\right) \text{ as } N \to \infty \tag{0.5}$$

for all $\mathbf{T} \in \overline{\mathcal{T}}_0$.

Aldous' result that after time 1/2, finite clusters always have the same law than critical cluster in Bernoulli percolation does play an important role in their proof of this result.

We will be to present what we believe to be a somewhat different proof based on differential inequalities rather than explicit expressions. Our approach seems more amenable to some modifications of the model. We will write this proof up here in the case of d-ary trees (connected graphs with no cycles with all vertices having degree d) for the particular choice of the good size function $\mathbf{T} \mapsto |\mathbf{T}|$, but the technique can be implemented in some other setups as well. (Note that the case $d = 4$ would be the one corresponding to alkane-type polymerization.) Let denote the law of the finite parameter percolation freezing clusters according to number of edges in the d-ary tree as $\mathbb{P}_N^d$.

Proposition 3. *For all $d \geq 3$, for all finite subtree $\mathbf{T}$ of the d-ary tree and all $t \leq 1$,*

$$\lim_{N\to\infty} \mathbb{P}_N^d(\mathcal{C}_t = C) = \mathbb{P}_\infty^d(\mathcal{C}_t = C). \tag{0.6}$$

Modified coalescence and freezing rules

In view to investigate in which sense the fine details of the coalescence and freezing rules matter for its definition and properties, it is also natural to wonder whether the finite-but-large freezing will also approximate well Brouwer's frozen site-percolation model.

The model with parameter $N \in \mathbb{N} \cup \{+\infty\}$ is described as follows: at any time t a vertex can be in one of three possible states, white (inactive), green (active), or red (frozen). Until the clock of a vertex rings, it is inactive (white). When the clock rings the vertex gets activated and turns green. However, as soon as this vertex belongs to a connected active component with size bigger than N

then the vertex and all the vertices in this active connected component change their state to red or frozen immediately (so it is possible that a vertex actually switches directly from white to red). Note that all white vertices do therefore end up being green or red. By time 1, when the process ends, we end up with a configuration of big frozen clusters with some small green clusters, all of which having size at most N. We denote by $\mathbf{P}_N$, for $N \in \mathbb{N} \cup \{\infty\}$ the law of the process described above.

Again, there is no definition problem for this model $\mathbf{P}_N$, and just as in Aldous' case, the asymptotic behavior when $N \to \infty$ for subcritical times $t \in [0, 1/2]$ is straightforward: The limiting picture is that of subcritical site percolation on the tree, and no site is red. The interesting question is about the behavior of the model when $t > 1/2$. As we have already mentioned, the fact that after time $1/2$ finite clusters had the same law than critical cluster in Bernoulli percolation played a key role in the proof of (3) in [12], so that some new ideas were needed to treat this site-percolation case.

One of the result of the present thesis is to show that the model with freezing of finite clusters does converge to Brouwer's frozen site-percolation model in the following sense:

Proposition 4. *Let v_0 be a distinguished vertex in V, and let $\mathcal{C}_t$ denote the green cluster of v_0 at time t. For $\mathbf{T}$ a finite subtree of $\mathbb{T}$ containing v_0 and all $t \in [0, 1]$,*

$$\mathbf{P}_N(\mathcal{C}_t = \mathbf{T}) \longrightarrow \mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}), \text{ as } N \to \infty. \tag{0.7}$$

One of the ideas of the proof is to circumvent the fact that the law of typical finite clusters when after the gelation time is no longer constant by using the aformentioned fact that the law of some functional of these clusters, namely the *tree of their interior vertices* (i.e., with degree 3) is actually stationary.

Conclusion and outlook

In conclusion, we hope that the present thesis will shed some new light on the frozen percolation models and their approximation through finite-range interacting particle systems. We believe that some of the techniques that we developed here are applicable to a rather large class of models and freezing rules. Of course, a number of natural open problems on frozen percolation models remain – one can for instance mention the frozen percolation questions in high-but-finite dimensional settings.

Chapter 1

Preliminaries

In this chapter, we briefly review two main classical themes about trees: In the first section, we briefly recall the basic classical features of percolation on Galton-Watson trees (for a reference see [45]), and we use this to introduce the notation that we will use throughout this thesis. In the second section, we briefly review the generating function techniques that allow to obtain enumerative formulas for trees, and we apply it to one specific case that we shall use later in this thesis, when we will discuss frozen site-percolation on the binary tree.

1.1 Percolation on Galton-Watson trees

The starting point of *percolation theory* was the work [18] of Broadbent and Hammersley in 1957. Percolation was introduced as a simple model for the way that a fluid propagates in porous media. To describe this model we consider a graph $G = (V, E)$, where V denotes its set of vertices and E its set of (unoriented) edges. For $H \subseteq G$ a subgraph of G we denote by $V(H)$ the subset of vertices of H and by $E(H) = \{(x,y) \in E : x, y \in V(H)\}$ the set of edges of H. Let p be a real number in $[0,1]$. The random process of *bond percolation* on G with parameter p is defined as follows: consider the space $\{0,1\}^E$. The elements $\omega = \{\omega(e)\}_{e\in E}$ of this space are called *configurations*. We say that an edge e is *open* in the configuration ω if $\omega(e) = 1$ and *closed* otherwise. The measure governing the percolation process with parameter p is the product measure P_p on $\{0,1\}^E$, where

$$P_p := (p\delta_1 + (1-p)\delta_0)^{\otimes E} . \tag{1.1}$$

Under this measure every edge in E is *open* with probability p and *closed* with probability $1-p$ independently of all other edges. To each configuration ω we associate the graph generated by the set of edges

$$\{e \in E : \omega(e) = 1\}. \tag{1.2}$$

We will not distinguish between a configuration ω and the graph generated by the set $\{e \in E : \omega(e) = 1\}$.

Percolation can be regarded as an increasing graph-valued process indexed by time t in $[0,1]$. Let $\{\tau_e\}_{e\in E}$ be an i.i.d. sequence of random variables distributed uniformly on $[0,1]$ under some probability measure P. We define the increasing process

$$\mathcal{B}_t(G) := \{e \in E : \tau_e \leq t\}, \tag{1.3}$$

for $t \in [0,1]$. In this setting we say that an edge e is open at time t if $e \in \mathcal{B}_t(G)$. By defining $\{\eta_t(e)\}_{e\in E}$ with $\eta_t(e) := \mathbb{1}_{\{\tau_e \leq t\}}$ we can write $\mathcal{B}_t(G)$ the set of open edges at time t as

$$\mathcal{B}_t(G) = \{e \in E : \eta_t(e) = 1\}. \tag{1.4}$$

From (1.4) it is clear that under P, for each $t \in [0,1]$ the graph generated by $\mathcal{B}_t(G)$ has the law P_t with P_t as in (1.1). Moreover, P yields a natural coupling in p (or t) of the measures P_p, that we will implicitly use throughout this thesis.

The analogous process can be defined for vertices by considering an i.i.d. sequence of uniform random variables in $[0,1]$ assigned to each vertex in the graph. In this case the process is known as *site percolation*.

We are interested in the connectivity properties of the sets (1.3). We say that two vertices u and v in V are adjacent if $(u, v) \in E$, in this case we write $u \sim v$. A path from u to v is a finite alternating sequence of vertices and edges $u_0, e_0, \ldots, u_{k-1}e_{k-1}u_k$, $k \in \mathbb{N}$, with $u_0 = u$, $u_k = v$ and $u_i \sim u_{i+1}$ for all $i \in \{0, \ldots, k-1\}$. Two vertices u and v in V are said to be connected in the percolation process at time t if they are connected by a path with all edges in $\mathcal{B}_t(G)$. We write this event as

$$\left\{ u \overset{\mathcal{B}_t(G)}{\longleftrightarrow} v \right\}. \tag{1.5}$$

With this notation we define the *open cluster* of a vertex u at time t as

$$B_t(u) := \{v \in V : u \overset{\mathcal{B}_t(G)}{\longleftrightarrow} v\}. \tag{1.6}$$

For a subgraph $H \subseteq G$ we define the size of H as $|H| := |E(H)|$, the cardinality of the set $E(H)$.

One of the main questions in percolation theory is: how does the distribution of the size of open clusters change as a function of t? In particular in an infinite graph, how does the probability of the event that a given vertex belong to an infinite open connected component change as t varies?

In the following we treat with graphs with a distinguished vertex O usually referred to as the root (or distinguished vertex) of G. We define in this case

$$\theta_G(t) := P\left(|B_t(O)| = \infty\right). \tag{1.7}$$

Because $\{\mathcal{B}_t(G)\}_{t \in [0,1]}$ is increasing in t, the function $\theta_G(\cdot)$ is also increasing in t and we can define

$$t_c(G) := \sup\{t \in [0, 1] : \theta_G(t) = 0\}. \tag{1.8}$$

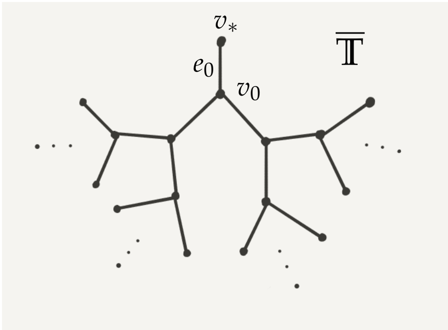

Figure 1.1: The tree $\overline{\mathbb{T}}$

This parameter is known as the *critical parameter* for percolation. The value $t_c(G)$ does not depend on the choice of the vertex O.

The above questions in percolation have a relatively easy answer when the underlying graph G is the d-regular tree, or more generally, a Galton-Watson tree.

Recall that a regular tree is an infinite connected graph with no cycles. It is called d-regular if every vertex on the graph has degree d, i.e. exactly d edges are incident on every vertex.

We denote the d-regular tree as $\mathbb{T}_d = (V_d, E_d)$ and we distinguish a vertex in V_d that we denote by O. We will often deal specifically with the binary tree ($d = 3$) and we reserve the notation $\mathbb{T}$ for $\mathbb{T}_3$. We define *the planted d-regular tree* as an infinite connected graph with no cycles where all vertices have degree d except for a vertex with degree one. We denote as $\overline{\mathbb{T}}_d = (\overline{V}_d, \overline{E}_d)$ the planted d-regular tree. The vertex with degree one in $\overline{\mathbb{T}}_d$ is denoted as v_*, the edge to which v_* is incident we denote it as e_0 and the second vertex incident to e_0 and different from v_* we denote it as v_0. With an slight abuse of notation we omit the dependency of this objects on d. Again we reserve the notation $\overline{\mathbb{T}}$ for $\overline{\mathbb{T}}_3$ (which is often also referred to as the binary tree).

A more general class of trees that we will consider are the Galton-Watson trees, that are defined as follows. One starts with one particle which has a random number of children according to an offspring distribution $\pi = (\pi_j)_{j\geq 0}$ (in other words, the number of offsprings is equal to j with probability π_j). Throughout this work, when we discuss Galton-Watson trees with offspring distribution π_j, then this will mean that $\pi_1 \neq 1$. Every descendant of the initial particle has progeny according to the same offspring distribution π independently of everything that has happened before. The genealogy tree resulting from this process is called a Galton-Watson tree.

If $\pi_0 \neq 0$, then with positive probability, the Galton-Watson tree can be finite (at some generation the progeny of a Galton-Watson tree gets extinguished). Let us list some very classical results on Galton-Watson trees:

- The Galton-Watson tree is almost surely finite if and only if $\overline{\pi} := \sum_j j\pi_j \leq 1$. In other words, the progeny of the root will have a positive probability to be infinite if and only if the expected number of offsprings of an individual is greater than 1.

- When $\pi_0 > 0$ and $\sum_j j\pi_j > 1$, then the probability that the progeny of the root is infinite is the unique solution $\theta \in (0,1)$ to the equation
$$1 - \theta = \sum_j \pi_j (1-\theta)^j.$$
This equation just reflects the fact that the root has a finite progeny if and only if all of its offspring have a finite progeny.

The analysis of percolation on Galton-Watson trees turns out to be easy. Here the model has two layers of randomness. First, one chooses the Galton-Watson tree according to an offspring distribution π, and then one performs percolation of parameter t on this Galton-Watson tree. The main observation is that the cluster that contains the root that one obtains in this way is again a Galton-

Watson tree, with offspring distribution π^t, given by

$$\pi^t(j) := \sum_{i\geq 0} \pi_i \frac{(i+j)!}{i!j!} t^j (1-t)^i.$$

This just reflects the fact that for an individual in this tree to have j descendants, then for some $i \geq 0$, it had $j+i$ descendants in the original Galton-Watson tree, and that out of these $j+i$ descendent edges, exactly j are open.

A special case is of course when π is the atomic mass at $d-1$, so that the initial Galton-Watson tree is the deterministic tree obtained from $\overline{\mathbb{T}}_d$ by removing the edge e_0. The obtained Galton-Watson tree has then a binomial offspring distribution.

Noting that that expected number of offsprings for π^t is

$$\overline{\pi}^t := t\overline{\pi},$$

we conclude immediately that when one performs percolation with parameter t in a Galton-Watson tree with offspring distribution π, the root will be in an infinite cluster with positive probability if and only if $t > 1/\overline{\pi}$. Furthermore, one can also write down the equation that characterizes the probability $\theta(t)$ for this cluster to be infinite, when $t > 1/\overline{\pi}$:

$$1-\theta(t) = \sum_{j\geq 0} \pi_j (1-t\theta(t))^j.$$

This is an "annealed" result (where one averages over both the random tree and the percolation process). However, a simple $0-1$ law argument can then be used in order to see that in fact, almost surely on the event where the Galton-Watson tree is infinite, the critical probability for percolation on that infinite tree is almost surely $1/\overline{\pi}$.

In the case of the d-regular tree, we therefore see that the critical probability is $t_c(d) := 1/(d-1)$. Furthermore, in that case, when $t > 1/(d-1)$,

$$\theta(t) = 1-(1-t\theta(t))^d. \tag{1.9}$$

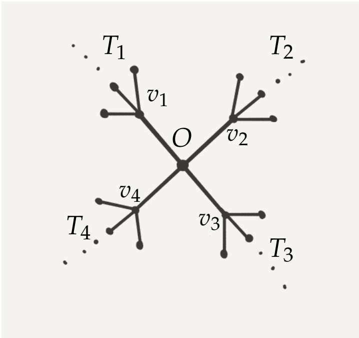

Figure 1.2: The tree $\mathbb{T}_4$ and T_1, T_2, T_3 and T_4

Besides the percolation probability, there are many features that are easy to describe for percolation on Galton-Watson trees. For instance, the actual law of the cluster containing the root edge e_0 in Bernoulli percolation on $\overline{\mathbb{T}}_d$ at time t is explicit: Let $\mathbf{T}$ be a finite subtree of $\overline{\mathbb{T}}_d$ containing e_0. We define the external edge boundary of $\mathbf{T}$ as

$$\partial_b \mathbf{T} := \left\{ e = (u,v) \in \overline{E}_d : u \in V(\mathbf{T}) \text{ and } v \in \overline{V}_d \setminus V(\mathbf{T}) \right\}. \tag{1.10}$$

We denote by B_t the open cluster containing e_0 at time t,

$$P(B_t = \mathbf{T}) = t^{|\mathbf{T}|}(1-t)^{|\partial_b \mathbf{T}|}. \tag{1.11}$$

The above law and the use of explicit enumeration results for d-regular trees then provide also an explicit law for the size of open clusters. Many other explicit computations can be done for percolation on regular or Galton-Watson trees due to its geometry, for example critical exponents can be explicitly and rigorously computed. For a reference in the particular case $d = 3$ see Chapter 10 in [35], or for general results on Galton-Watson trees see [45]. For a reference on percolation on Galton-Watson trees seen as a tree-valued Markov chain see for example [1] or [8].

1.2 Simple combinatorics on d-regular trees

We now review the classical combinatorial techniques based on generating functions (see for instance [24] and [33]) used to enumerate certain trees. We present these results here as they will be used later in the thesis.

For G a tree-graph, we say that $\mathbf{T}$ is a subtree of G if it is a connected subgraph of G. Finite subtrees of $\overline{\mathbb{T}}_d$ containing the edge e_0 are called d-Catalan trees. In this section we give some enumerative properties of these families of trees. We first give details on how to obtain enumerative formulae for the case $d = 3$.

Define

$$\mathcal{T}_k := \{\mathbf{T} \text{ subtrees of } \overline{\mathbb{T}} \text{ containing } e_0 \text{ with } |\mathbf{T}| = k\}, \tag{1.12}$$

for $k \geq 0$, with the convention that $\mathcal{T}_0$ is the subset containing just the graph consisting of the vertex v_* (with no edges). This convention results practical for counting purposes.

It is a classical result in combinatorics that the cardinality of the set $\mathcal{T}_k$ for $k \geq 0$ is given by the Catalan numbers

$$c_k := \binom{2k}{k}\frac{1}{k+1}. \tag{1.13}$$

At studying enumerative properties of big combinatorial objects like trees one very useful tool is the use of generating functions. These ones provide a way to encode combinatorial properties of the objects in a nice language that facilitate their study. Many generating functions come from recurrence formulas arising from the in-

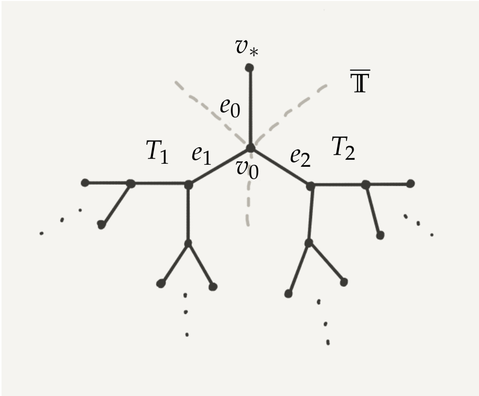

Figure 1.3: Recursive decomposition of $\overline{\mathbb{T}}$

variance of the objects that we are trying to understand as in the case of d-Catalan trees. We review the ideas of how to obtain the explicit expressions provided in (1.13) for c_k by using the counting generating function associated to the sequence $\{c_k\}_{k\geq 0}$. We present this classical result with the purpose of getting familiar with these methods that are going to be used later in the study of particular subclasses of Catalan trees.

We define the generating function associated to $\{c_k\}_{k\geq 0}$ as

$$C(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{1.14}$$

for $z \in \mathbb{R}$. Consider the tree $\overline{\mathbb{T}}$ and denote by e_1 and e_2 the adjacent edges to v_0 which do not contain the vertex v_* as an end vertex. We denote the two neighboring vertices of v_0 distinct from v_* by v_1 and v_2 respectively. We denote by T_1 and T_2 the two infinite subtrees of $\overline{\mathbb{T}}$ isomorphic to $\overline{\mathbb{T}}$ which contain the edges e_1 and e_2 respectively with $V(T_1) \cap V(T_2) = \{v_0\}$. See Figure 1.3.

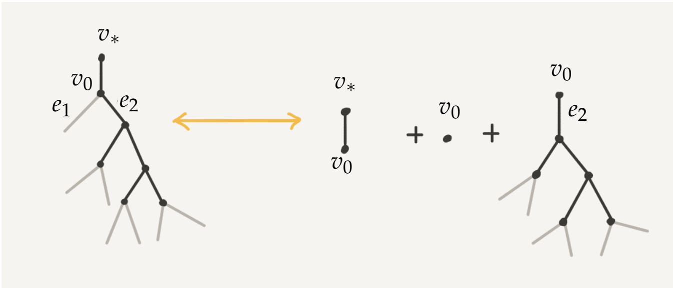

Figure 1.4: The decomposition of a tree in $\mathcal{T}_6$

We define for $k \geq 0$,

(1.15)
$$\mathcal{T}_k^1 := \{\mathbf{T} \text{ subtrees of } T_1 \text{ containing the edge } e_1 \text{ with } |\mathbf{T}| = k\},$$

and

(1.16)
$$\mathcal{T}_k^2 := \{\mathbf{T} \text{ subtrees of } T_2 \text{ containing the edge } e_2 \text{ with } |\mathbf{T}| = k\}.$$

We use the same convention as for $\mathcal{T}_0$ that $\mathcal{T}_0^1$ and $\mathcal{T}_0^2$ are the subsets containing just the graph consisting of the vertex v_1 and v_2 respectively with no edges.

Fix $k \geq 1$, every subtree $\mathbf{T}$ in $\mathcal{T}_k$ can be decomposed in a unique way in to the edge e_0 and two subtrees (being possible the trivial tree with just the vertex v_0 and no edges) $\mathbf{T}_1$ in $\mathcal{T}_i^1$ and $\mathbf{T}_2$ in $\mathcal{T}_j^2$ with $i, j \geq 0$ satisfying $i + j = k - 1$. The subtrees $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined as follows: if $\mathbf{T}$ is the only element of $\mathcal{T}_1$ define $\mathbf{T}_1$ and $\mathbf{T}_2$ as the only elements of the sets $\mathcal{T}_0^1$ and $\mathcal{T}_0^2$ respectively. Otherwise, define $\mathbf{T}_1$ as follows. If e_1 is not in $E(\mathbf{T})$ then set $\mathbf{T}_1$ to be the only element of $\mathcal{T}_0^1$, otherwise remove the edges e_0 and e_2 from $\mathbf{T}$, after this operation we end up with two connected components of $\mathbf{T}$, set $\mathbf{T}_1$ to be the component containing e_1. The definition of $\mathbf{T}_2$ is analogous. On the other way around, merging through the vertex v_0 two subtrees in $\mathcal{T}_i^1$ and $\mathcal{T}_j^2$ with $i + j = k - 1$ give rise a unique element of $\mathcal{T}_k$ and all elements of $\mathcal{T}_k$ are obtained in this way, see Figure 1.4, it might be helpful at this point. Therefore we see that for $k \geq 1$, the elements of $\mathcal{T}_k$ and $\bigcup_{\substack{i,j\geq 0\\ i+j=k-1}} \mathcal{T}_i^1 \times \mathcal{T}_j^2$ are in a bijective relation that leads to the recursion

$$c_k = \sum_{\substack{i,j\geq 0\\ i+j=k-1}} c_i c_j \tag{1.17}$$

for $k \geq 1$. Multiplying both sides in (1.17) by z^k and summing over $k \geq 0$ we obtain the equation for C

$$C(z) = 1 + zC^2(z). \tag{1.18}$$

Solving the quadratic equation for $C(z)$ we obtain that

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z} = \frac{2}{1 + \sqrt{1-4z}}, \tag{1.19}$$

which for $z \in [-1/4, 1/4]$ has the series expansion

$$C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{k+1} z^k \tag{1.20}$$

and then (1.13) follows.

Let us now consider the generalization of the sequence c_n for d-regular trees. For each $d \geq 3$ we define the d-Catalan numbers

$$c_k^d := \binom{k(d-1)}{k} \frac{1}{k(d-2)+1}, \tag{1.21}$$

for all $k \geq 0$. The connected subgraphs of $\overline{\mathbb{T}}_d$ containing e_0 having k edges are enumerated by the d-Catalan numbers (see for reference [38], [51]): for all $d \geq 3$ and for all $k \geq 0$,

$$c_k^d = \left|\{\mathbf{T} \text{ subtrees of } \overline{\mathbb{T}}_d \text{ containing } e_0 \text{ with } |\mathbf{T}| = k\}\right|. \tag{1.22}$$

Indeed, the sequence c_k^d (as well as the right-hand side of the equation here) is easily seen to satisfy the recursive formula

$$c_k^d = \sum_{\substack{i_1,\dots,i_{d-1}\geq 0,\\ i_1+\dots+i_{d-1}=k-1}} c_{i_1}^d \dots c_{i_{d-1}}^d, \tag{1.23}$$

For all $d \geq 3$, one can also define the generating function associated to d-Catalan numbers for all $z \in \mathbb{R}$ as,

$$C_d(z) := \sum_{k=0}^{\infty} c_k^d z^k, \tag{1.24}$$

and note that it satisfies the function equation

$$C_d(z) = 1 + z(C_d(z))^{d-1}.$$

For notational purposes, we will also use the notation C^d for C_d.

In order to determine the radius of convergence of the generating function in (1.24) we can use Stirling's formula to see that, for each $d \geq 3$,

$$\binom{k(d-1)}{k} \frac{1}{k(d-2)+1} \sim \sqrt{\frac{d-1}{2\pi}} \frac{1}{(k(d-2))^{3/2}} \left(\frac{(d-1)^{d-1}}{(d-2)^{d-2}}\right)^k, \tag{1.25}$$

as $k \to \infty$. This shows that the radius of convergence of the power series C_d is

$$R_d := \frac{1}{(d-1)} \left(\frac{d-2}{d-1}\right)^{d-2}, \ d \geq 3. \tag{1.26}$$

The following result will also be useful:

Lemma 5. *For all $d \geq 3$,*

$$C_d(R_d) = \frac{d-1}{d-2}. \tag{1.27}$$

Here is a brief probabilistic proof of this simple fact.

Proof. Let us consider critical Bernoulli percolation on the tree $\overline{\mathbb{T}}_d$. Recall that the critical probability in that case is $t_c = 1/(d-1)$, and that almost surely, there is no infinite cluster at this time. This means that if we sum over all possible finite trees $\mathbf{T}$ the probability that the cluster that contains the origin is equal to $\mathbf{T}$, we get 1. We regrouping all trees according to their size, we get that

$$1 = \frac{d-2}{d-1} + \sum_{k\geq 1} c_k^d \left(\frac{1}{d-1}\right)^k \left(\frac{d-2}{d-1}\right)^{k(d-2)+1}.$$

Rewriting the last line we get that

$$1 = \frac{d-2}{d-1}\left(1 + \sum_{k=1}^{\infty} c_k \left(\frac{(d-2)^{d-2}}{(d-1)^{(d-1)}}\right)^k\right) \tag{1.28}$$

which proves the claim. □

To conclude this part we define the partial sums for $z \geq 0$

$$C_N^d(z) := \sum_{k=0}^{N} c_k^d z^k. \tag{1.29}$$

For each fix $d \geq 3$ the function $C_N^d(z)$ is an strictly increasing function on z and $C_N^d(z) \to \infty$ as $z \to \infty$. This observation allows us to define the sequence of roots $\{w_N^d\}_{N\geq 1}$, where w_N^d is the only positive root of $C_N^d(z) - C_d(R_d) = 0$. Note that $\{w_N^d\}_{N\geq 1}$ is a convergent sequence since it is decreasing and bounded from below. Moreover by Lemma 5,

$$w_\infty^d := \lim_{N\to\infty} w_N^d = R_d. \tag{1.30}$$

1.3 Some variations

Throughout this section, we will work in the binary tree, and we will collect some information that will be useful later in our analysis of frozen site-percolation. We define the infinite tree $\widehat{\mathbb{T}}$ as the tree obtained from $\overline{\mathbb{T}}$ by removing the vertex v_* and its incident edge e_0, we root this tree on the vertex v_0.

1.3.1 Counting trees according to vertex-type

For $\mathbf{T}$ a subtree of $\widehat{\mathbb{T}}$ we define the internal vertex boundary of $\mathbf{T}$

$$\partial\mathbf{T} := \{v \in V(\mathbf{T}) : \exists u \in V \setminus V(\mathbf{T}) \text{ with } v \sim u\}. \tag{1.31}$$

We observe that in $\partial\mathbf{T}$ there are two kind of vertices: the ones having exactly one neihgbour in the complement of $V(\mathbf{T})$, which we call *vertices of type I*, and the ones having two, called *vertices of type II*, see Figure 1.5. In the model of modified frozen percolation, vertices of type I and vertices of type II on the boundary of an active cluster do not play the same role. For this reason it is important to be able to determine how many different subtrees of $\widehat{\mathbb{T}}$ exists containing v_0 and having a given number of vertex of type I and type II. For a subtree $\mathbf{T}$ of $\widehat{\mathbb{T}}$ we denote the set of vertices of type I and the set of vertices of type II of $\mathbf{T}$ by $\partial_1\mathbf{T}$ and $\partial_2\mathbf{T}$ respectively. We define the set $\mathcal{I}(\mathbf{T}) := V(\mathbf{T}) \setminus \partial\mathbf{T}$, the subset of *internal vertices* of $\mathbf{T}$. We denote the number of internal vertices of $\mathbf{T}$ as $I(\mathbf{T}) := |\mathcal{I}(\mathbf{T})|$, and the number of vertices of type I of $\mathbf{T}$ as $J(\mathbf{T}) := |\partial_1\mathbf{T}|$. We see that the number of vertices of type II and the total size of $\mathbf{T}$ is determined by $I(\mathbf{T})$ and $J(\mathbf{T})$. Indeed, to see this define the external vertex boundary of a subtree $\mathbf{T}$ of $\widehat{\mathbb{T}}$ as

$$\widetilde{\partial}\mathbf{T} := \{v \in V \setminus V(\mathbf{T}) : \exists u \in V(\mathbf{T}) \text{ such that } u \sim v\}. \tag{1.32}$$

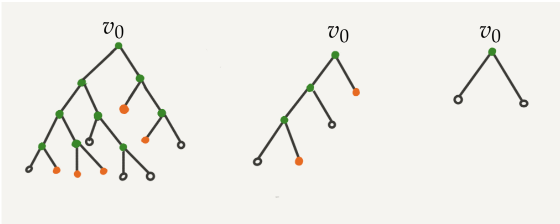

Figure 1.5: Three different configurations in the modified frozen percolation on $\widehat{\mathbb{T}}$. On the left a configuration where the vertex v_0 is an internal vertex. In the middle a configuration where v_0 is of type I. On the right a configuration where v_0 is of type II.

Note that

$$|\widetilde{\partial}\mathbf{T}| = 2|\partial_2\mathbf{T}| + |\partial_1\mathbf{T}|, \tag{1.33}$$

and that

$$|\widetilde{\partial}\mathbf{T}| = |V(\mathbf{T})| + 1. \tag{1.34}$$

Furthermore,

$$|V(\mathbf{T})| = |\mathcal{I}(\mathbf{T})| + |\partial_1(\mathbf{T})| + |\partial_2(\mathbf{T})|. \tag{1.35}$$

From (1.33), (1.34) and (1.35) together it follows that

$$|\partial_2\mathbf{T}| = I(\mathbf{T}) + 1 \tag{1.36}$$

and that

$$|V(\mathbf{T})| = 2I(\mathbf{T}) + J(\mathbf{T}) + 1. \tag{1.37}$$

Then, knowing the number of vertices of type I and the number of internal vertices is enough to determine the size of a tree and the number of vertices of type II.

We determine now the number of elements in the set

$$\mathcal{T}_{ij} := \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} \text{ containing } v_0 \text{ with } I(\mathbf{T}) = i, J(\mathbf{T}) = j\} \tag{1.38}$$

for $i, j \geq 0$ and we denote

$$a_{ij} := |\mathcal{T}_{ij}|. \tag{1.39}$$

For $x, y \in \mathbb{R}$ we define the bivariate generating function associated to the sequence $\{a_{ij}\}_{i,j\geq 0}$,

$$F(x,y) := \sum_{i,j\geq 0} a_{ij} x^i y^j. \tag{1.40}$$

We use a similar decomposition as the one we used in the last section to get a recursion formula for the numbers a_{ij}. We denote by v_1 and v_2 the two neighbor vertices of v_0 and we denote by $\widehat{T}_1$ and $\widehat{T}_2$ the two graphs isomorphic to $\widehat{\mathbb{T}}$ containing v_1 and v_2 respectively, that are obtained when we remove the vertex v_0 and the two edges incident to it. See figure 1.6.

In order to give a recursive relation for the numbers a_{ij} we distinguish among three types of trees in $\mathcal{T}_{ij}$ according to the type of v_0 (internal, type I or type II). This classification of the trees induces a partition of $\mathcal{T}_{ij}$ for all $i, j \geq 0$. When $i, j = 0$ the vertex v_0 is necessarily of type II and then the set $\mathcal{T}_{00}$ has only one element, the graph consisting just of the vertex v_0 (with no edges). We consider this graph as a subtree of $\widehat{\mathbb{T}}$ and we call this graph the trivial tree and we denote it as $\mathbf{T}_0$. With this convention the set $\mathcal{T}_{00}$ is the set containing just the tree $\mathbf{T}_0$. Thus we obtain the boundary condition

$$a_{00} = 1. \tag{1.41}$$

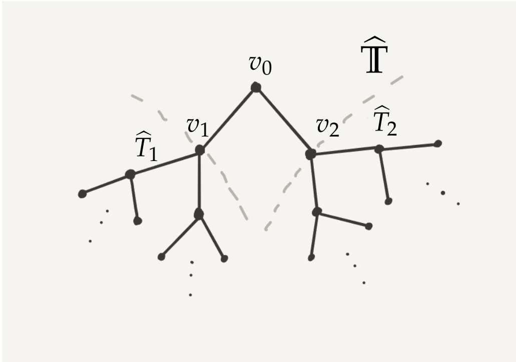

Figure 1.6: The decomposition of $\widehat{\mathbb{T}}_0$ in to $\widehat{T}_1$ and $\widehat{T}_2$.

When $i + j \geq 1$, $\mathcal{T}_{ij}$ is the disjoint union of the sets

$$\mathcal{T}_{ij} \cap \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} : v_0 \text{ is of type I }\} \tag{1.42}$$

and

$$\mathcal{T}_{ij} \cap \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} : v_0 \text{ is an internal vertex}\}. \tag{1.43}$$

We look separately to the cardinality of these two subsets of $\mathcal{T}_{i,j}$. We consider first the cardinality of the set in (1.42).

For a subtree $\mathbf{T}$ in $\mathcal{T}_{ij}$, with v_0 being a vertex of type I we can further distinguish among two cases: the only neighbor of v_0 belongs to either $\widehat{T}_1$ or belongs to $\widehat{T}_2$. If we suppose that it is the former case, we can decompose in a unique way the tree $\mathbf{T}$ in to the edge containing (v_0, v_1) and a subtree $\mathbf{T}_1$ of $\widehat{T}_1$ containing v_1 with $j-1$ vertices of type I and i internal vertices. The subtree $\mathbf{T}_1$ is defined as the tree obtained from $\mathbf{T}$ by removing the edge (v_0, v_1), see Figure 1.7. On the other direction any subtree of $\widehat{T}_1$ with i internal vertices and $j-1$ vertices of type I produces a unique element of the set in (1.42) by merging it with the edge (v_0, v_1) through the vertex v_1. Then, there is a bijective relation between the elements in the set in (1.42)

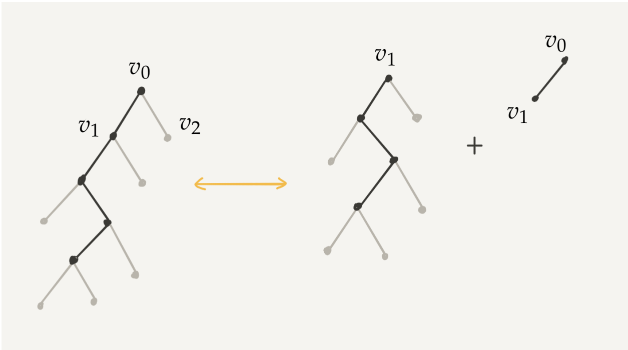

Figure 1.7: The decomposition of a subtree of $\widehat{\mathbb{T}}$ with four vertices of type I and no internal vertices. The vertex v_0 is of type I and has a neighbor in $\widehat{T}_1$.

and the set

$$(1.44)\qquad \{\mathbf{T} \text{ subtree of } \widehat{T}_1 \text{ containing } v_1 : I(\mathbf{T}) = i, J(\mathbf{T}) = j-1\}.$$

Since $\widehat{T}_1$ is isomorphic to $\widehat{\mathbb{T}}$, the previous observations lead to the relation for $j \geq 1$ and $i \geq 0$

$$(1.45)\qquad |\mathcal{T}_{ij} \cap \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} : v_0 \text{ is of type I}, v_1 \in V(\mathbf{T})\}| = a_{ij-1}.$$

Analogously it holds that

$$(1.46)\qquad |\mathcal{T}_{ij} \cap \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} : v_0 \text{ is of type I}, v_2 \in V(\mathbf{T})\}| = a_{ij-1}.$$

Thus, we conclude from (1.45) and (1.46) that for $j \geq 1$

$$(1.47)\qquad |\mathcal{T}_{ij} \cap \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} : v_0 \text{ is of type I }\}| = 2a_{i,j-1}.$$

For (1.43), observe that a tree $\mathbf{T}$ in $\mathcal{T}_{ij}$ with v_0 being an internal vertex can be decomposed in to the tree consisting of v_0 together with its two adjacent edges and two subtrees $\mathbf{T}_1 \subset \widehat{T}_1$ and $\mathbf{T}_2 \subset \widehat{T}_2$ having each of them i' and i'' internal vertices and j' and j'' vertices of type I respectively and satisfying $i' + i'' = i - 1$ and $j' + j'' = j$. The two subtrees $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined as follows: remove from the tree $\mathbf{T}$ the edges (v_0, v_1) and (v_0, v_2), after that two connected components of $\mathbf{T}$ are left, define $\mathbf{T}_1$ as the component containing the vertex v_1 and define $\mathbf{T}_2$ as the component containing v_2, see Figure 1.8. On the other way around any tree in the set (1.43) can be constructed by merging through the vertices v_1 and v_2 the subtree containing v_0 with three vertices and v_0 being an internal vertex (the left-most element on the right of Figure 1.8) and two subtrees of $\widehat{T}^1$ and $\widehat{T}_2$ with the previous considerations on the number of internal vertices and vertices of type I. Therefore we see that the elements in the set in (1.43) and in the set

$$\bigcup_{\substack{i',i'',j',j''\geq 0\\ i'+i''=i-1, j'+j''=j}} \mathcal{T}^1_{i'j'} \times \mathcal{T}^2_{i''j''}, \tag{1.48}$$

with

$$\mathcal{T}^1_{ij} := \{\mathbf{T} \text{ subtree of } \widehat{T}_1 \text{ containing } v_1 : I(\mathbf{T}) = i, J(\mathbf{T}) = j\} \tag{1.49}$$

and

$$\mathcal{T}^2_{ij} := \{\mathbf{T} \text{ subtree of } \widehat{T}_2 \text{ containing } v_2 : I(\mathbf{T}) = i, J(\mathbf{T}) = j\}, \tag{1.50}$$

are in a bijective relation. More over since $\widehat{T}_1$ and $\widehat{T}_2$ are both isomorphic to $\widehat{\mathbb{T}}$, we see that the cardinality of the set in (1.48) is given for $i \geq 1, j \geq 0$ by

$$\sum_{\substack{i',i''\geq 0\\ i'+i''=i-1}} \sum_{\substack{j',j''\geq 0\\ j'+j''=j}} a_{i',j'} a_{i'',j''}. \tag{1.51}$$

We thus get from (1.47) and (1.51) that for $i, j \geq 1$,

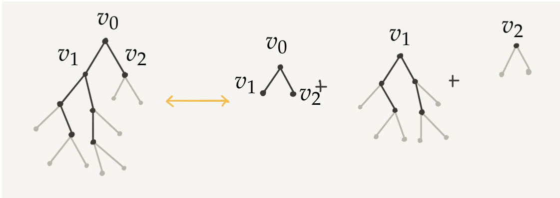

Figure 1.8: The decomposition of a subtree $\mathbf{T}$ where v_0 is an internal vertex

(1.52) $$a_{i,j} = 2a_{i,j-1} + \sum_{\substack{i',i''\geq 0\\ i'+i''=i-1}} \sum_{\substack{j',j''\geq 0\\ j'+j''=j}} a_{i',j'}a_{i'',j''}.$$

Multiplying both sides in the above equation by x^i and y^j, summing over i and j, and using (1.41) we get

(1.53)
$$F(x,y) = 1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} 2a_{i,j-1}x^i y^j + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \sum_{\substack{i',i''\geq 0\\ i'+i''=i-1}} \sum_{\substack{j',j''\geq 0\\ j'+j''=j}} a_{i',j'}a_{i'',j''}x^i y^j.$$

In order to get to a nicer expression for F we define the set

(1.54) $$\mathcal{H} := \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} \text{ containing } v_0 : \mathcal{I}(\mathbf{T}) = \emptyset\},$$

the set of finite subtrees of $\widehat{\mathbb{T}}$ containing v_0 with no internal vertices. Note that $\mathcal{H}$ contains the tree $\mathbf{T}_0$. The generating function H associated to the set $\mathcal{H}$ is defined as

(1.55) $$H(y) := \sum_{j=0}^{\infty} a_{0,j} y^j.$$

From (1.47), the function H satisfies the equation

(1.56) $$H(y) = 1 + 2yH(y),$$

which has solution

(1.57) $$H(y) = \frac{1}{1-2y}.$$

For y with $|y| < 1/2$ the function H admits a power expansion as,

(1.58) $$H(y) = \sum_{j=0}^{\infty} (2y)^j.$$

(Note that from the above display we can conclude that $a_{0,j} = 2^j$ for $j \geq 0$). Using (1.55), we can rewrite the first summand in the right side of (1.53) as

(1.59) $$1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} 2a_{i,j-1}x^i y^j = H(y) + 2y\left(F(x,y) - H(y)\right),$$

and the second one as $xF(x,y)^2$. We thus obtain the quadratic equation for the generating function F,

(1.60) $$F(x,y) = H(y) + 2y\left(F(x,y) - H(y)\right) + xF(x,y)^2.$$

Solving this quadratic equation for F, we get

(1.61) $$F(x,y) = \left(\frac{1-2y}{2x}\right)\left(1 - \sqrt{1 - \frac{4x}{(1-2y)^2}}\right).$$

Note that F has region of convergence $\mathcal{R} := \{(x,y) \in \mathbb{R}^2 : |\frac{4x}{(1-2y)^2}| \leq$

$1/4\}$. By making the series expansion for F in the region $\mathcal{R}$, we obtain that

$$a_{i,j} = 2^i \frac{(2i+j+1)!}{i!(i+1)!j!}. \tag{1.62}$$

To conclude this part we define the function

$$G(x,y) := xF(x,y) \tag{1.63}$$

for $(x,y) \in \mathcal{R}$. Later it would be more convenient to work with the function G instead of working with F.

1.3.2 A transformation on trees

In this part we introduce a transformation on trees that plays a key role for us in the study of the modified percolation process on $\mathbb{T}$.

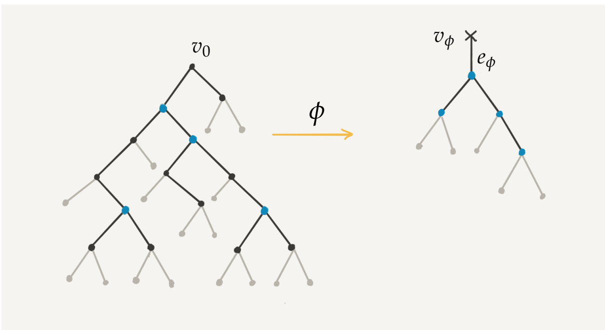

Figure 1.9: On the left a subtree $\mathbf{T}$ with 4 internal vertices marked in blue. On the right $\phi(\mathbf{T})$

Let $\mathbf{T}$ be a subtree in $\mathcal{T}_{ij}$, $i,j \geq 0$. We define the following transformation ϕ on the tree $\mathbf{T}$, we describe the transformation on three steps: first, we add to $\mathbf{T}$ an artificial edge e_ϕ incident to v_0 and to a new vertex v_ϕ. The second step on the transformation consist in contracting any sequence of vertices in $J(\mathbf{T})$ to a single edge. In the last step, we remove the remaining vertices of type II and the edges incident to them. Note that at the end of this operation on $\mathbf{T}$ we end up with a tree $\phi(\mathbf{T})$ that is isomorphic to a subtree in $\mathcal{T}_i$ as in (1.12). In particular, if $\mathbf{T}$ has no internal vertices then $\phi(\mathbf{T})$ is the tree consisting of the edge e_ϕ.

We note that any tree $\mathbf{T} \in \mathcal{T}_{i,j\geq 0}$ can be encoded by the tree $\phi(\mathbf{T})$ together with $i+1$ subtrees of $\mathbf{T}$. More precisely, we define the set

$$\mathcal{D} := \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} \text{ containing } v_0 : \mathcal{I}(\mathbf{T}) = \{v_0\}\}, \tag{1.64}$$

and we recall the definition of the set $\mathcal{H}$ introduced before in (1.54).

A tree $\mathbf{T}$ in $\mathcal{T}_{ij}$ we encode it by $\phi(\mathbf{T})$ and $i+1$ trees $\mathbf{T}_\phi, \mathbf{T}_1, \ldots, \mathbf{T}_i$, each of these trees corresponding with one vertex of $\phi(\mathbf{T})$. The tree $\mathbf{T}_\phi$ is isomorphic to an element of $\mathcal{H}$ and each of the trees $\mathbf{T}_1, \ldots, \mathbf{T}_i$ is isomorphic to an element of $\mathcal{D}$. To define the tree $\mathbf{T}_\phi$, if $v_0 \in \mathcal{I}(\mathbf{T})$ we set $\mathbf{T}_\phi = \mathbf{T}_0$ the trivial tree, if not denote by $\tilde{e}$ the last edge in the path from v_0 to the first internal vertex of $\mathbf{T}$. By removing $\tilde{e}$ from $\mathbf{T}$ we end up with two connected components of $\mathbf{T}$, let $\mathbf{T}_\phi$ be the component containing v_0. The trees $\mathbf{T}_1, \ldots, \mathbf{T}_i$ correspond each of them to a vertex in $V(\phi(\mathbf{T})) \setminus \{v_\phi\}$. If $i \geq 1$, denote by v_1 the neighbor vertex of v_ϕ on $\phi(\mathbf{T})$ i.e., $e_\phi = (v_\phi, v_1)$. Enumerate the rest of vertices in $\phi(\mathbf{T})$ in any order $\{v_2, \ldots, v_i\}$. Each of the vertices $v_1, \ldots, v_i$ correspond to exactly one internal vertex of $\mathbf{T}$, denote the corresponding internal vertex to v_j in $\mathbf{T}$ as $\phi^{-1}(v_j)$. Let $\mathbf{T}_j$, $j \in \{1, \ldots, i\}$ be the tree rooted at the internal vertex in $\mathbf{T}$, $\phi^{-1}(v_j)$ obtained in the following way: remove from $\mathbf{T}$ the edge incident to $\phi^{-1}(v_j)$ in the unique path from $\phi^{-1}(v_j)$ to v_0, keep the connected component containing the vertex $\phi^{-1}(v_j)$. If in this connected component there are no more internal vertices, set $\mathbf{T}_j$ to be this subtree. Otherwise consider all paths from internal vertices to $\phi^{-1}(v_j)$ in this connected component, remove the edges incident to internal vertices which do not lie in these paths, this action produces different connected components, set $\mathbf{T}_j$ to be the one containing the vertex $\phi^{-1}(v_j)$.

Now we explain how to recover $\mathbf{T}$ from the trees $\phi(\mathbf{T})$ and $\mathbf{T}_\phi$, $\mathbf{T}_1, \dots, \mathbf{T}_i$. The tree $\phi(\mathbf{T})$ encodes the adjacencies between the trees $\mathbf{T}_\phi, \mathbf{T}_1, \dots, \mathbf{T}_i$. If $\mathbf{T}_\phi = \mathbf{T}_0$ we identify the vertices v_ϕ and the vertex v_1. Otherwise we link the trees $\mathbf{T}_\phi$ and $\mathbf{T}_1$ by adding an edge between the root vertex of both trees. The trees corresponding to adjacent vertices in $\phi(\mathbf{T})$ are merged together in the following way: consider two vertices adjacent on $\phi(\mathbf{T})$ v_j and $v_{j'}$. Assume without loss of generality that the vertex v_j is closer to the vertex v_ϕ. Suppose that $v_{j'}$ is the right child of v_j. The tree $\mathbf{T}_j$ is rooted at an internal vertex and has two vertices of type II each of them in a different line of descent (right and left), we merge the trees $\mathbf{T}_j$ and $\mathbf{T}_{j'}$ by joining the root vertex of $\mathbf{T}_{j'}$ to the vertex of type II of $\mathbf{T}_j$ on the right side of descent of $\mathbf{T}_j$. In the case that $v_{j'}$ is the left child of v_j the procedure is analogous. See figure (1.10).

We finish this section with an observation on the generating function F that is very natural after considering the transformation on trees ϕ and the encoding of trees described before.

We define the set

$$\mathcal{E} := \{\mathbf{T} \text{ subtree of } \widehat{\mathbb{T}} \text{ containing } v_0 : v_0 \in \mathcal{I}(\mathbf{T})\}. \tag{1.65}$$

and let E be the generating function associated to the set $\mathcal{E}$.

Composition is a combinatorial operation that consists in replacing certain sub-configurations by those of a second kind in all possible ways to obtain a third set of distinct configurations where any element in the third subset is uniquely constructed in this way.

Through the encoding of trees given before we can see that $\mathcal{E}$ is obtained from the composition of the set $\bigcup_{i\geq 0} \mathcal{T}_i$ and the set $\mathcal{D}$. We can construct all trees in $\mathcal{E} \cap \mathcal{T}_{ij}$ by replacing each of the vertices of an element in $\mathcal{T}_i$ by an element of $\mathcal{D}$.

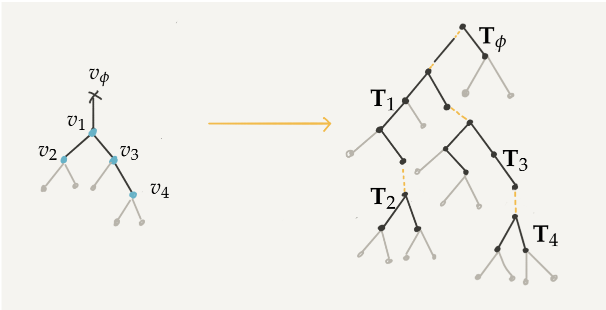

Figure 1.10: The tree on the left of Figure 1.9 can be encoded by the tree $\phi(\mathbf{T})$ on the left, and the trees $\mathbf{T}_\phi, \mathbf{T}_1, \ldots, \mathbf{T}_4$ on the right

The construction of $\mathcal{E}$ as the composition of the sets $\bigcup_{i\geq 0} \mathcal{T}_i$ and $\mathcal{D}$ has consequences for the generating function associated to $\mathcal{E}$.

By a recursive decomposition similar to the ones used before to obtain equation (1.60) it is easy to see that the generating function of the set $\mathcal{D}$ is equal to

$$D(x,y) = xH(y)^2. \tag{1.66}$$

Recall that the set $\bigcup_{i\geq 0} \mathcal{T}_i$ has associated the generating function C as in (1.14). The Composition Lemma (see Lemma 2.2.22 in [33] for a reference) implies that

$$E(x,y) = C(D(x,y)) = C(xH(y)^2). \tag{1.67}$$

This together with the fact that any tree in $\bigcup_{i,j\geq 0} \mathcal{T}_{ij}$ can be obtained as the product of an element in $\mathcal{E}$ and one in $\mathcal{H}$ implies that we can rewrite $F(x,y)$ in terms of the Catalan generating function and the function H as (1.19) as

$$F(x,y) = H(y)C\left(xH(y)^2\right). \tag{1.68}$$

Using the series expansion of H and C we get by (1.68) that

$$a_{ij} = c_i \binom{2i+j}{j} 2^j, \tag{1.69}$$

where c_i is as in (1.13).

In view of the encoding of finite trees $\mathbf{T}$ containing v_0 through $\phi(\mathbf{T})$ and $\mathbf{T}_\phi, \ldots, \mathbf{T}_{|\phi(\mathbf{T})|}$ it is easy to understand (1.69). The number of trees with j vertices of type I that give rise to the same tree of internal vertices with i edges is 2^j times the number of ways to distribute the vertices of type I in to the trees $\mathbf{T}_\phi, \ldots, \mathbf{T}_i$. There are $\binom{2i+j}{j}$ ways to do this. The factor 2^j comes from choosing for each vertex of type I a direction, each vertex of type I has a neighbor on the tree on its left or on its right. Then we see that there are $\binom{2i+j}{j}2^j$ trees with j vertices of type I that give rise to the same tree of internal vertices whereas there are c_i possible configurations of i internal vertices.

We could arrive directly to either (1.68) or (1.69) by just making simple computations for (1.61) and (1.62), but it is through the views of the transformation ϕ that we find a way to have a better understanding of the geometry of subtrees of $\widehat{\mathbb{T}}$ and the mechanism of modified frozen percolation on $\mathbb{T}$.

Chapter 2

Frozen percolation processes on trees

In this chapter we will discuss the existence of the model of frozen percolation in infinite trees, as defined in [6] by Aldous in regular trees. The chapter is split in four sections. In the first section we review the results and arguments by Aldous in [6], who did write up the details in the case of the tree where vertices have all degree 3. Aldous also pointed out in that paper that his arguments were also applicable in the case of the binary tree. Since it will be relevant in the next chapters, we will write up the details also in that case here. We then point out that some of the arguments also go through in order to make sense of the frozen (bond) percolation process on Galton-Watson trees. We also briefly review Brouwer's analogous construction of frozen percolation model on the binary tree, where this time, percolation and freezing is with respect to *site* percolation. Finally, we very briefly recall Benjamini and Schramm's argument that shows that a frozen percolation process can not exist on the infinite square lattice.

2.1 Aldous' existence result on d-regular trees

We start this section by defining the dynamics of the model of frozen percolation on d-regular trees in detail. We fix $d \geq 3$. To avoid heavy notation, we omit must of the dependencies of functions and processes on d, we will make clear the dependence just when there exists a risk of confusion.

Let $\{\tau_e\}_{e\in E_d}$ be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$. Set $\mathcal{A}_0 = \emptyset$.
(2.1)
For every edge $e = (u,v) \in E_d$, at time $t = \tau_e$ if the connected components containing the vertices u and v in $\mathcal{A}_{t^-}$ are finite we set

$$\mathcal{A}_t = \mathcal{A}_{t^-} \cup \{e\},$$

otherwise we set $\mathcal{A}_t = \mathcal{A}_{t^-}$.

In words, every edge has assigned a random clock with uniform distribution on the time interval $[0,1]$. At time 0 all edges are closed, when the clock assigned to an edge rings, this one gets open in the frozen percolation process if its both end vertices belong to finite components, other wise the edge stays closed forever. At time one, the set of open edges in the frozen percolation process is a subgraph of $\mathbb{T}_d$. There are infinite clusters surrounded by frozen edges that never got to open, and small (finite) clusters separated from infinite components by frozen edges.

The definition of the model seems natural and simple but preventing big clusters to keep growing makes harder to study the process as in comparison with ordinary percolation on $\mathbb{T}_d$. The difficulty starts already with the existence of the process, a process with the dynamics above described is not even clear to exists. The process cannot be defined in any infinite graph as Bernoulli percolation, $\mathbb{Z}^2$ is such case. This remark was done by Benjamini and Schramm, [10]. As we were not able to find a reference with the details of this remark we include at the end of this chapter a sketch for the proof of this fact.

To a large degree, the difficulty in the study of the model comes from the apparent absence of a simple criteria in terms of the random variables $\{\tau_e\}_{e\in E_d}$ to describe the process. This is in constrast with the rather similarly defined process $\mathcal{D} = \{\mathcal{D}_t\}_{t\in[0,1]}$ with the following dynamics. At time 0, we set $\mathcal{D}_0 = \emptyset$. Every edge $e = (u,v) \in E_d$ joins the process at time $t = \tau_e$ if and only if at least one of its end vertices belongs to a finite component of $\mathcal{D}_{t^-}$. This process is a lot easier to study because the criteria for an edge to join the process can be expressed easily in terms of the i.i.d. clocks $\{\tau_e\}_{e\in E_d}$: an edge e joins the process at time τ_e if and only if at least one of its vertices belongs to a finite cluster of the set $\{e \in E_d : \tau_e \leq t\}$. This process is considered in [36] and it is related to the Minimal Essential Spanning Forest of regular trees. Apparently there is no such simple criteria to define $\mathcal{A}_t$ in terms of $\mathcal{B}_t(\mathbb{T}_d)$, recall that $\mathcal{B}(\mathbb{T}_d)$ is the Bernoulli percolation process defined as

(2.2) $$\mathcal{B}_t(\mathbb{T}_d) = \{e \in E_d : \tau_e \leq t\}, \qquad t \in [0,1]$$

in Section 1.1. In the following we omit the dependency of the process $\mathcal{B}$ on the underlying graph as this will be clear from the context.

The processes $\mathcal{A}$ and $\mathcal{B}$ can be coupled by means of the collection of random variables $\{\tau_e\}_{e\in E_d}$ in a natural way. Under this coupling a.s. for all $t \in [0,1]$,

(2.3) $$\mathcal{A}_t \subseteq \mathcal{B}_t,$$

actually the process $\mathcal{A}_t$ coincides with the process $\mathcal{B}_t$ until the critical time $t_c(\mathbb{T}_d)$. This relation between the two processes implies immediately existence of the process $\mathcal{A}$ until the critical time. The relation $\mathcal{B}_t = \mathcal{A}_t$, $t \in [0, t_c(\mathbb{T}_d)]$ holds because up to time $t_c(\mathbb{T}_d)$ there are no infinite clusters in $\mathcal{B}_t(\mathbb{T}_d)$ a.s., therefore, due to (2.3), also a.s. for all times before $t_c(\mathbb{T}_d)$ all clusters in $\mathcal{A}_t$ are finite, all edges with their clock ringing before time t_c get to open also in the frozen percolation process and the existence of the process $\mathcal{A}$ up to time t_c is clear.

We turn to explain the ideas of Aldous to define the process on the whole time interval $[0,1]$. The general argument used in there is named in [7] "the 540° argument" (one and a half circles, the reason will be clear in the following paragraph) and it is applied in other settings (see [7] and [9] for a reference) where a Recursive Distributional Equation (RDE) is associated to a tree-index process.
In general the method comprises three steps: first assume the existence of a process satisfying some qualitative properties and do heuristics that lead to an RDE. As a second step solve the RDE, and construct the process rigorously using the distribution found by solving the RDE. To end prove that the constructed process satisfies the qualitative properties that were assumed at the beginning.
In the particular case of frozen percolation: first assume that the process exists on the planted tree $\overline{\mathbb{T}}_d$ and make natural assumptions on independence and invariance of the process. This assumptions lead to an RDE for the distribution of the time at which the edge e_0 gets frozen, i.e. the time at which e_0 joins an infinity connected component in the frozen percolation process. Solve the RDE by solving a differential equation, use the solution found in this way to define rigorously a joint law on directed edges on $\mathbb{T}_d$ and use this law to defined the process. Finally prove that the model defined in this way satisfies the initial assumption that infinite clusters get frozen (edges neighboring infinite clusters can not open).

We review the details of this procedure done in [6] and we extend at the same time the proofs for the d-regular case.

Theorem 6. *(Theorem 1 in [6], Aldous 2000) For all $d \geq 3$, there exists a joint law for $\mathcal{A}$ and $\{\tau_e\}_{e \in E_d}$ such that under this law the process $\mathcal{A}$ satify the dynamics described in* (2.1) *and the joint law is invariant under automorphisms of $\mathbb{T}_d$.*

We begin the construction assuming existence of the model on the planted d-regular tree $\overline{\mathbb{T}}_d$ and doing some heuristics. Denote by $e_1, \ldots, e_{d-1}$ the incident edges to the vertex v_0 in $\overline{\mathbb{T}}_d$ different from e_0. As in the previous section, for the case $d = 3$, we denote by $T_1, \ldots, T_{d-1}$ the subtrees of $\overline{\mathbb{T}}_d$ isomorphic to $\overline{\mathbb{T}}_d$, containing e_1, e_2, $\ldots$, e_{d-1} respectively and whose intersection of vertices is $V(T_1) \cap$

$\cdots \cap V(T_{d-1}) = \{v_0\}$, see Figure 1.3 for the case $d = 3$. In what follows the critical time $t_c(\mathbb{T}_d) = (d-1)^{-1}$ will appear often in explanations and proofs, to save some notation we use from now on the notation $t_c(d) := t_c(\mathbb{T}_d)$.

We define Y the random time at which e_0 gets frozen in the frozen percolation process defined on $\overline{\mathbb{T}}_d$. Due to the absence of infinite clusters in the process up to time $t_c(d)$, the random variable Y takes values on $[t_c(d), 1] \cup \{\infty\}$. The value ∞ is taken in the case that e_0 never joins a giant cluster. Analogously to Y we define $Y_1, \ldots, Y_{d-1}$ the random times at which the edges $e_1, \ldots, e_{d-1}$ get frozen in the percolation processes defined on the trees T_1, T_2, $\ldots$, T_{d-1} respectively. Because of invariance of the process, since all T_i, $i \in \{1, \ldots, d-1\}$ are isomorphic to $\overline{\mathbb{T}}_d$, the frozen percolation processes on the trees T_i's have all the same law of the frozen percolation process on $\overline{\mathbb{T}}_d$. Furthermore they are independent since the processes defined on $T_1, \ldots, T_{d-1}$ depend on disjoint sets of edges. Due to these two reasons (invariance and independence) the random variables $Y, Y_1, \ldots, Y_{d-1}$ have the same distribution and $\{Y_i\}_{1 \le i \le d-1}$ is a family of independent random variables. Moreover by the dynamics described in (2.1) it needs to hold that if τ_{e_0} is greater or equal than $\min\{Y_1, \ldots, Y_{d-1}\}$, then the edge e_0 never gets to be open and in particular $Y = \infty$. If on the contrary, $\tau_{e_0} < \min\{Y_1, \ldots, Y_{d-1}\}$, then the edge e_0 will join an infinite cluster at the time

$$Y = \min\{Y_1, \ldots, Y_{d-1}\}.$$

We define the function $\varphi : [t_c(d), 1] \cup \{\infty\} \times [0, 1] \to [t_c(d), 1] \cup \{\infty\}$,

$$\varphi(z, u) := \begin{cases} z & \text{if } u < z \\ \infty & \text{if } z \le u. \end{cases} \tag{2.4}$$

The behaviour of Y described in the paragraph above corresponds to say that

$$Y = \varphi(\min\{Y_1, \ldots, Y_{d-1}\}, \tau), \tag{2.5}$$

where τ is a random variable distributed uniformly on $[0, 1]$ independent of $\{Y_i\}_{1 \leq i \leq d-1}$. The equality in (2.5) establishes an RDE for the common distribution of the random variables $Y_1, \ldots, Y_{d-1}$. We denote by ν this common law.

For the case $d = 3$, Aldous proved that there is only one distribution supported on $[t_c(d), 1] \cup \{\infty\}$ satisfying the recursion formula in (2.5). To be more precise:

(2.6)
let $Y_1, \cdots, Y_{d-1}$ be independent random variables with distribution according to a probability measure μ and let τ be a random variable independent of $\{Y_i\}_{1 \leq i \leq d-1}$ with uniform distribution on $[0, 1]$, we say that the distribution μ is a solution of the recursive distributional equation (2.5) if

$$\varphi(\min\{Y_1, \ldots, Y_{d-1}\}, \tau),$$

with φ as in (2.4), is distributed according to μ.

Lemma 7 (Lemma 3 , Aldous [6]). *A non-atomic distribution μ on $[t_c(d), 1]$ is a solution to the recursive distributional equation* (2.6) *if and only if there exists $t_0 \in [t_c(d), 1]$ such that*

$$\mu(dt) = \frac{d-1}{d-2}(t(d-1))^{-(d-1)/(d-2)} dt, \qquad t_c(d) \leq t \leq t_0 \tag{2.7}$$

and

$$\mu(\infty) = \left(\frac{1}{t_0(d-1)} \right)^{1/(d-2)}. \tag{2.8}$$

Proof. Assume that μ is a solution to the RDE (2.6). We denote by $F(t) := \mu[0, t]$ the distribution function of μ. Let $Y_1, \ldots, Y_{d-1}$ be i.i.d. random variables with common distribution function F and let τ

be a random variable independent of $\{Y_i\}_{1\le i\le d-1}$ having uniform distribution on $[0,1]$.
The fact that the distribution μ is a solution of the RDE (2.6) implies

$$F(t) = \mathbb{P}(\varphi(\min\{Y_1,\dots,Y_{d-1},\tau\}) \le t). \tag{2.9}$$

According to the definition of φ in (2.4) the above equation holds if and only if

$$F(t) = \mathbb{P}\left(\tau < \min\{Y_1,\dots,Y_{d-1}\} \le t\right). \tag{2.10}$$

Conditioning with τ on the right side of the above equality we get

$$F(t) = \int_0^t \mathbb{P}\left(s < \min\{Y_1,\dots,Y_{d-1}\} \le t\right) ds. \tag{2.11}$$

Using that $Y_1,\dots,Y_{d-1}$ are i.i.d. random variables we get that the integral in the above display is equal to

$$F(t) = \int_0^t \left((1-F(s))^{d-1} - (1-F(t))^{d-1}\right) ds. \tag{2.12}$$

Taking the derivative in both sides of (2.12) gets us to the differential equation for F,

$$F'(t) = tF'(t)(d-1)\left(1-F(t)\right)^{d-2}, \tag{2.13}$$

for all $t \in [t_c(d),1]$, which imply

$$F(t) = 1 - \frac{1}{(t(d-1))^{1/(d-2)}} \tag{2.14}$$

on $[t_c(d),1] \cap$ support (μ). Note that the function $t \mapsto 1-(t(d-1))^{\frac{-1}{d-2}}$ is strictly increasing on $[t_c(d),1]$, this holds only if support (μ) is a closed interval of the form $[t_c(d),t_0]$ for some $t_c(d) \le t_0 \le 1$. The proof of the other direction is straight forward. $\square$

The above lemma implies that the distribution ν is given by,

$$\nu(dt) = \frac{d-1}{d-2}(t(d-1))^{-(d-1)/(d-2)}dt, \qquad t_c(d) \le t \le 1 \tag{2.15}$$

and

$$\nu(\infty) = \left(\frac{1}{d-1}\right)^{1/(d-2)}. \tag{2.16}$$

Now we go back to the tree $\mathbb{T}_d$ to define the process satisfying the desired dynamics. Denote by $\vec{E}_d$ the set of directed edges of $\mathbb{T}_d$,

$$\vec{E}_d := \{\text{ordered pairs } \langle u, v\rangle : u, v \in V_d\}. \tag{2.17}$$

There is a natural genealogy relation on directed edges, a directed edge $\vec{f}$ is said to be the child of edge $\vec{e}$ if the end vertex of $\vec{e}$ is the origin vertex of $\vec{f}$.

With the aid of Lemma 7 we define a process on the set of directed edges of $\mathbb{T}_d$.

Lemma 8. *(Lemma 4 in [6], Aldous 2000) There exists a joint law for $\{(Y_{\vec{e}}, \tau_{\vec{e}})\}_{\vec{e}\in\vec{E}_d}$ invariant under graph automorphisms on $\mathbb{T}_d$ and such that for all $\vec{e} \in \vec{E}_d$ the random variable $Y_{\vec{e}}$ has law ν and*

$$Y_{\vec{e}} = \varphi(\min\{Y_{\vec{e}_1}, \dots, Y_{e_{\vec{d-1}}}\}, \tau_{\vec{e}}),\ a.s. \tag{2.18}$$

where $\vec{e}_1, \dots, \vec{e}_{d-1}$ are the children of the edge $\vec{e}$.

Proof. Define the random variables $\tau_{\vec{e}} := \tau_e$ for $\vec{e} \in \vec{E}_d$ any direction of the edge e.

Fix any edge $\tilde{e} \in E_d$. We are going to define the law of $Y_{\vec{e}}$ in increasing neighborhoods of $\tilde{e}$. The length of a path between two vertices u and v is the number of edges in the path. We define

$$\begin{aligned} V_d^{\le h} := \{v \in V_d : &\text{ the minimal distance from } v \\ &\text{to a vertex of } \tilde{e} \text{ is at most } h\}, \end{aligned} \tag{2.19}$$

we define $\vec{E}_d^{\leq h}$ to be the set of directed edges generated from the vertices in $V_d^{\leq h}$, that is

$$\vec{E}_d^{\leq h} := \{\vec{e} = \langle u, v \rangle \in \vec{E}_d : u, v \in V_d^{\leq h}\}. \tag{2.20}$$

and let $\vec{E}_d^h$ be the set of directed edges in $\vec{E}_d$ at distance exactly h from e which are directed away from e, more precisely

$$\begin{aligned} \vec{E}_d^h := \{\vec{e} = \langle u, v \rangle \in \vec{E}_d^{\leq h} \setminus \vec{E}_d^{\leq h-1} : \vec{e} \text{ is} \\ \text{the child of an edge in } \in E_d^{\leq h-1}\}. \end{aligned} \tag{2.21}$$

Assign i.i.d. random variables $\{Y_{\vec{e}}\}_{\vec{e} \in \vec{E}_d^h}$ independently from $\{\tau_{\vec{e}}\}_{e \in \vec{E}}$ to the edges in $\vec{E}_d^h$ with distribution as in (2.7). The directions of edges in $E_d^{\leq h}$ allow to use the recursion (2.5) to define the rest of random variables $\{Y_{\vec{e}}\}_{e \in E_d^{\leq h}}$ for all edges in $E_d^{\leq h}$.
All $Y_{\vec{e}}$ for $\vec{e} \in \vec{E}_d^{\leq h}$ have law ν due to Lemma 7. This implies that for all $h \in \mathbb{N}$ the laws defined on the sets $\vec{E}_d^{\leq h}$ are consistent. Therefore, by Kolmogorov extension theorem, there exists a joint law of $\{(Y_{\vec{e}}, \tau_{\vec{e}})\}_{\vec{e} \in \vec{E}_d}$ satisfying that $\{\tau_{\vec{e}}\}_{\vec{e} \in \vec{E}_d}$ are i.i.d uniform random variables on $[0, 1]$ and all $Y_{\vec{e}}$ for all $\vec{e} \in \vec{E}_d$ satisfy (2.18). It is clear that the law with the finite-dimensional distributions described above is invariant under transformations that preserve the connective relations on the graph $\mathbb{T}_d$. $\square$

Once the existence of the joint law for $\{(Y_{\vec{e}}, \tau_{\vec{e}})\}_{\vec{e} \in \vec{E}_d}$ is proved we define

$$\mathcal{A}_1 = \left\{ e \in E_d : \tau_e < \min\{Y_{\vec{e}_1}, \dots, Y_{\vec{e}_{2(d-1)}}\} \right\}, \tag{2.22}$$

where $\vec{e}_1 \dots, \vec{e}_{2(d-1)}$ are the $2(d-1)$ directed edges in $\vec{E}_d$ that are directed away from e. For $0 \leq t < 1$ we define,

$$\mathcal{A}_t := \{e \in \mathcal{A}_1 : \tau_e \leq t\}. \tag{2.23}$$

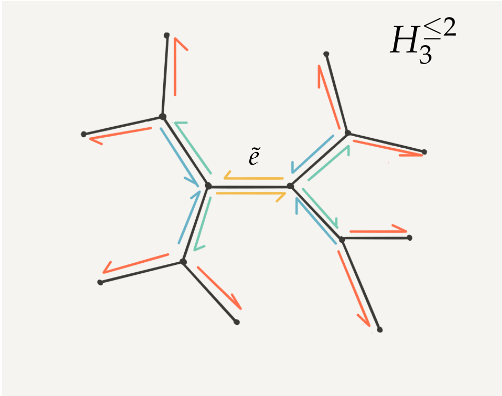

Figure 2.1: Definition of random variables $\{Y_{\vec{e}}\}_{\vec{e}\in \vec{E}_3^{\leq 2}}$ for $\vec{e} \in H_3^{\leq 2}$. First the random variables $\{Y_{\vec{e}}\}_{\vec{e}\in \vec{E}_3^2}$ are assigned to directed orange edges. After, the random variables $Y_{\vec{e}}$ are defined in the order: green, yellow, blue , using the recursion in (2.6).

From now on we denote by $\mathbb{P}_\infty$ the law of the process of frozen percolation constructed in this way. Aldous proved that the process defined in (2.22) and in (2.23) satisfies (2.1), to see that it is enough to prove:

Proposition 9. *(Proposition 6 in [6], Aldous 2000) Let $t < 1$. A vertex v percolates at time t if and only if $t > \min\{Y_{\vec{e}_1}, Y_{\vec{e}_2}, Y_{\vec{e}_3}\}$, where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the edges at v directed away from v.*

Thanks to the above proposition we can translate properties in terms of the dynamics described in (2.1) to properties of the random variables $\{Y_{\vec{e}}\}_{\vec{e}\in \vec{E}_d}$ and vice-versa . This will be useful in Section 3.1.

2.2 Frozen percolation in Galton-Watson trees

Throughout this section, we are going to consider a distribution $\pi := (\pi_j)_{j\geq 0}$ on the non-negative integers, that will represent the off-spring distribution of a Galton-Watson tree. We will assume that the expectation of the number of offsprings $\overline{\pi} := \sum_{j\geq 0} j\pi_j$ is greater than 1, so that this Galton-Watson tree has a positive probability to be infinite. We could also assume that $\pi_0 \neq 0$ if we wanted to ensure that the Galton-Watson tree is almost surely infinite, but this is not necessary.

When one performs bond percolation with parameter t on this Galton-Watson tree, then the distribution of the cluster containing the root is clearly again a Galton-Watson tree, with a new off-spring distribution that will have mean $t\overline{\pi}$. So, it follows immediately that the critical value for percolation on a Galton-Watson tree (or rather on a Galton-Watson tree conditioned to be infinite) will be $t_c := 1/\overline{\pi}$.

We now briefly explain that it is possible to generalize Aldous' construction of the frozen process on such trees. This should not be so surprising, because the construction in the case of d-ary trees was based on the recursive discovery of the tree, which works for Galton-Watson trees as well.

Since most arguments are identical to the ones for the d-ary tree, we will here only emphasize the points where things are slightly different.

We consider the Galton-Watson tree with offspring distribution π, and let e_0 denote the additional edge that is attached to the root. So, we are here in the setup of the "planted" tree with an oriented root edge e_0. Now, following Aldous' analysis, we assume that the frozen percolation process on such a Galton-Watson tree exists, and we define $F(t)$ to be the annealed (i.e., averaged also on the possible random trees – this is important in our analysis) probability that e_0 is frozen at time t. We note that $F(t) = 0$ for $t \leq t_c$, because of the absence of infinite clusters for such values of t.

One short heuristic way to express the arguments in the previous section, is to say that:

- The quantity $F(t)$ is necessarily positive for $t > t_c$ (otherwise, there are no frozen clusters, which would contradict the existence of infinite clusters for percolation for such values of t). The quantity $F(t)$ is furthermore a non-decreasing function of

t.

- When $t > t_c$, for e_0 to become frozen during the infinitesimal interval $[t, t+dt]$, the edge e_0 has to be opened before time t, and if one looks at the j descendent edges of e_0 and the j independent freezing processes corresponding to their descendent trees, none of these edges is frozen at time t and one of them should freeze during the time-interval $[t, t+dt]$.

This leads to the equation:

$$F(t+dt) - F(t) = (F(t+dt) - F(t)) \times t \times \sum_{j \geq 0} j\pi_j (1 - F(t))^{j-1}$$

(the j terms comes from the j possible choices among the descendent edges to decide where the freezing comes from). In other words,

$$\sum_{j \geq 1} j\pi_j (1 - F(t))^{j-1} = t^{-1}. \tag{2.24}$$

Since $t_c = 1/\overline{\pi}$, one can rewrite this as:

$$\sum_{j \geq 1} j\pi_j (1 - (1 - F(t))^{j-1}) = t_c^{-1} - t^{-1}, \tag{2.25}$$

for $t > t_c$.

Then, one can retro-engineer the analysis, and check that for each $t > t_c$, there exists a unique solution $F(t) \in [0,1]$ to the equation (2.25), and then use this solution $(F(t))_{t \geq 0}$ to actually construct the frozen percolation process.

As a sanity check, we see that we recover the value of the function for the d-ary tree when $a_j = 1_{j=d-1}$ and $t > 1/(d-1)$:

$$t^{-1} = (d-1)(1 - F(t))^{d-2},$$

i.e.,

$$1 - F(t) = 1/(t(d-1))^{1/(d-2)}$$

which is indeed the Aldous' formula.

Also, we see that the formulas for Galton-Watson trees with the following offspring distributions are quite nice:

- When the tree is a Galton-Watson tree with geometric off-spring distribution
$$\pi_j = (1-u)u^j$$
for $j \geq 0$ (and some given $u \in (1/2, 1)$ so that the tree has a positive probability to be infinite), then when $t > t_c$
$$t^{-1} = \sum j(1-u)uu^{j-1}(1-F(t))^{j-1} = \frac{u(1-u)}{(1-u(1-F(t)))^2}.$$
In other words, for all $t > t_c$,
$$(1-u+uF(t))^2 = tu(1-u),$$
and in particular when $t = 1$,
$$F(1) = \frac{\sqrt{u(1-u)} - (1-u)}{u}.$$
- Similarly, if we consider instead the Galton-Watson tree with geometric off-spring distribution conditioned to be non-zero given by $\pi_0 = 0$ and
$$\pi_j = (1-u)u^{j-1}$$
for $j \geq 1$ (and some given $u \in (0,1)$), then for all $t > t_c$,
$$(1-u+uF(t))^2 = t(1-u),$$
and in particular when $t = 1$,
$$F(1) = \frac{\sqrt{1-u} - (1-u)}{u}.$$
- Another nice case is when the number of offspring distribu-

tion is a Poisson distribution:

$$\pi_j = \lambda^j e^{-\lambda}/j!$$

for $j \geq 0$ (and some given $\lambda > 1$, so that the tree has a positive probability to be infinite). Then, $t_c = 1/\lambda$ and for $t > t_c$,

$$t^{-1} = \lambda e^{-\lambda} \sum_{j \geq 1} \frac{\lambda^{j-1}(1 - F(t))^{j-1}}{(j-1)!} = \lambda e^{-\lambda F(t)}.$$

In other words,

$$F(t) = \lambda^{-1} \log(t\lambda)$$

and in particular,

$$F(1) = \lambda^{-1} \log(\lambda).$$

- If one considers an offspring distribution with $\pi_j \neq 0$ only for $j = 1, 2$: Suppose that $\pi_2 = u$ and $\pi_1 = 1 - u$ for some $u \in (0, 1]$, then, $t_c = (1 + u)^{-1}$ and for $t > t_c$,

$$t^{-1} = (1 - u) + 2u(1 - F(t)) = 1 + u - 2uF(t),$$

and $F(1) = 1/2$ which does not depend on u.

2.3 Frozen site-percolation on the binary tree

We define the dynamics of the process introduced by Rachel Brouwer [19] where freezing on the 3-regular tree occurs according to site-percolation rules rather than bond percolation. Recall of course that when no freezing is considered, site percolation on a tree (even when it is not regular) is almost the same as bond percolation. One can for instance start with site percolation on a rooted tree. There is a simple bijection between the edges of the tree and the set of vertices (minus the root ρ): To each vertex $x \neq \rho$, associate the last edge $e(x)$ of the unique path from ρ to x. We can then relate a bond percolation to a site percolation model in the tree, by declaring that the edge $e(x)$ is open (in the bond-percolation model) if and only the site x is open (in the site-percolation model). Existence of an infinite cluster for the site-percolation model is equivalent to the existence of an infinite cluster for the bond-percolation model (as they both correspond the existence of infinite open rays). However, there are small marginal differences on the shapes of the clusters. For instance, in this correspondence, two distinct site percolation clusters could correspond to the same infinite bond percolation cluster). This suggests that the frozen percolation for site percolation will differ from the bond percolation one. Let us now describe Brouwer's model:

We start by defining in detail the dynamics of the process. Let $\{\tau_v\}_{v \in V}$ be an i.i.d. sequence of random variables uniformly distributed on $[0,1]$. The dynamics of the modified frozen percolation model on the 3-regular tree is described as follows: at any time t a vertex can be in one of three possible states, white, green, or red. Defining $\mathcal{W}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$ to be the sets of white, green, and red vertices at time t respectively. We set $\mathcal{W}_0 := V$, $\mathcal{R}_0 := \{\emptyset\}$ and $\mathcal{G}_0 := \{\emptyset\}$. We can think of the set $\mathcal{W}_t$ as the set of inactive vertices at time t, $\mathcal{G}_t$ the open/active vertices and $\mathcal{R}_t$ as the set of frozen vertices at time t. For $t \in [0,1]$ we denote by $\mathcal{C}_t(v)$ the connected subgraph of $\mathbb{T}$ containing v induced by the vertices in $\mathcal{G}_t \cup \{v\}$. For each $v \in V$ and $t = \tau_v$, we set

$$\mathcal{W}_t = \mathcal{W}_{t^-} \setminus \{v\}$$

and
(2.26)
If $|\mathcal{C}_{t^-}(v)| < \infty$, then we set

$$\mathcal{G}_t = \mathcal{G}_{t^-} \cup \{v\} \text{ and } \mathcal{R}_t = \mathcal{R}_{t^-}.$$

Otherwise we set

$$\mathcal{G}_t = \mathcal{G}_{t^-} \setminus \{V(\mathcal{C}_{t^-}(v))\} \text{ and } \mathcal{R}_t = \mathcal{R}_{t^-} \cup \{V(\mathcal{C}_{t^-}(v))\}.$$

In words, at time 0 all vertices are white (inactive), as time passes clusters of green (open/active) vertices appear and grow until their volume reaches infinite size. As soon as this happens they become red (freeze). The white vertices neighboring red clusters possibly become green at a later time. Hence the model allows for these vertices to freeze afterwards. By time 1, when the process ends, we end up with a configuration of big frozen clusters separated by small green clusters.

The model of modified frozen percolation has mainly two differences with respect to Aldous model: the first one is that in order to determine the occurrence of the event $\{\mathcal{C}_t = \mathbf{T}\}$, is not enough to look at the states of sites on the boundary of $\mathbf{T}$ just at time t but at all times in the whole time interval $[0, t]$. This change is caused by the change of the dynamics by allowing sites neighboring frozen clusters to open. The second difference is due to considering frozen percolation on sites. The vertices on the boundary of $\mathcal{C}_t$ with two neighbors in the external boundary plays a different role than vertices with one neighbor on the external boundary.

To conclude the introduction of this model we point out that the model on sites is a lot nicer to study because when one considers the modified version of Aldous model on edges the independence on the model that lead in that case to an RDE and nice formulas does not hold anymore. See Remark 6.2 in [19] for more details.

In [19], Brouwer followed Aldous' "540°-method" to prove existence of the modified frozen percolation model on the 3-regular tree. Assuming the existence of the model on $\widehat{\mathbb{T}}$ the planted vertex 3-regular tree, the natural assumptions of invariance and independence of the process lead to an RDE for the time at which v_0 gets frozen on the modified percolation process on $\widehat{\mathbb{T}}$. All the details of this construction are presented in [19].

Let v_1 and v_2 be the neighoboring vertices of v_0 in $\widehat{\mathbb{T}}$. Denote by $\widehat{T}^1$ and by $\widehat{T}^2$ the isomorphic trees to $\widehat{\mathbb{T}}$ rooted at the vertices v_1 and v_2 respectively that are obtained by removing v_0 and its two incident edges from $\widehat{\mathbb{T}}$. We have done such decomposition on Chapter 1, see Figure 1.6. Let $\widehat{Y}$ be the random time at which the vertex v_0 gets frozen on the modified percolation process on $\widehat{\mathbb{T}}$. Analogously

we define $\widehat{Y}_1$ and $\widehat{Y}_2$ the times at which the vertices v_1 and v_2 get frozen in their respectively processes on $\widehat{T}^1$ and $\widehat{T}^2$. The natural assumptions of independence and invariance on the model imply that $\widehat{Y}$, $\widehat{Y}_1$ and $\widehat{Y}_2$ have all the same distribution and $\widehat{Y}_1$ is independent from $\widehat{Y}_2$.
The possibility for vertices to get open despite being neighboring an infinite cluster makes to consider one more possibility as in Aldous case: If v_0 gets activated before the time at which v_1 and v_2 join an infinite component then v_0 will join an infinite component at the time $\min\{\widehat{Y}_1, \widehat{Y}_2\}$, if v_0 gets activated after the time at which exactly one of the neighboring vertices v_1 or v_2 got frozen, then $\widehat{Y} = \max\{\widehat{Y}_1, \widehat{Y}_2\}$. On the case that v_0 gets activated after $\max\{\widehat{Y}_1, \widehat{Y}_2\}$, v_0 will remain open until the end of the process at time 1, in particular will never join an infinite cluster and in this case $\widehat{Y}$ takes the value ∞. This behaviour can be written up in terms of the function $\widehat{\varphi} : [\frac{1}{2}, 1] \cup \{\infty\} \times [\frac{1}{2}, 1] \cup \{\infty\} \times [0, 1] \to [\frac{1}{2}, 1] \cup \{\infty\}$,

$$\widehat{\varphi}(x, y, u) := \begin{cases} x & \text{if } u < x < y \\ y & \text{if } x \leq u < y \\ \infty & \text{if } y \leq u, \end{cases} \tag{2.27}$$

as

$$\widehat{Y} = \widehat{\varphi}(\min\{\widehat{Y}_1, \widehat{Y}_2\}, \max\{\widehat{Y}_1, \widehat{Y}_2\}, \tau_{v_0}). \tag{2.28}$$

In [19], Brouwer proved that the only distribution $\widehat{\nu}$ on $[1/2, 1] \cup \{\infty\} \times [1/2, 1] \cup \{\infty\} \times [0, 1]$ with no atoms and strictly increasing on $[1/2, 1]$ is given by

$$\begin{aligned} \widehat{\nu}(dt) &= \frac{1}{2t} dt, \qquad \text{for } t \in [1/2, 1], \\ \widehat{\nu}(\infty) &= 1 - \ln(2). \end{aligned} \tag{2.29}$$

The proof is very similar to the proof of Lemma 7.

Lemma 10. *(Lemma 3.1 in [19], Brouwer 2005) Let μ be a probability distribution function, on $[1/2, 1] \cup \{\infty\}$ with no atoms on $[1/2, 1]$. Let $(\widehat{Y}_1, \widehat{Y}_2, \tau)$ be independent random variables, $\widehat{Y}_1$, $\widehat{Y}_2$ each having probability distribution function μ and τ having the uniform distribution on $[0, 1]$. Then,*

$$\widehat{\varphi}(\min\{\widehat{Y}_1, \widehat{Y}_2\}, \max\{\widehat{Y}_1, \widehat{Y}_2\}, \tau), \tag{2.30}$$

has again probability distribution μ, if and only if μ is defined as in (2.29).

Proof. Denote by $\widehat{F}$ the distribution function of μ. According to the definition of $\widehat{\varphi}$, the random variable $\widehat{\varphi}(\min\{\widehat{Y}_1, \widehat{Y}_2\}, \max\{\widehat{Y}_1, \widehat{Y}_2\}, \tau)$ is distributed according to μ if and only if

$$\widehat{F}(t) = t[1 - \left(1 - \widehat{F}(t)\right)^2] - 2(1 - \widehat{F}(t)) \int_0^t \widehat{F}(s) - \int_0^t \widehat{F}(s)^2 ds. \tag{2.31}$$

Taking derivatives in the above equation we thus get,

$$\widehat{F}'(t) = \widehat{F}'(t)[2t(1 - \widehat{F}(t))] + 2\widehat{F}'(t) \int_0^t \widehat{F}(s)ds, \tag{2.32}$$

for $t \in [t_c, 1] \cap \mathrm{support}(\mu)$ this implies,

$$1 = [2t(1 - \widehat{F}(t))] + 2 \int_0^t \widehat{F}(s)ds, \tag{2.33}$$

differentiating a second time we get

$$\widehat{F}'(t) = \frac{1}{t} \tag{2.34}$$

with the boundary condition $\widehat{F}(1/2) = 0$, we conclude that

$$\widehat{F}(t) = \ln(2t) \tag{2.35}$$

on $[1/2, 1] \cap \mathrm{support}\ \mu$. Since the function on (2.35) is strictly increasing on $[1/2, 1]$ and the distribution μ has no atoms on $[1/2, 1]$

this implies that the support of $\widehat{\nu}$ is the closed interval $[1/2, 1]$.

The other direction on the statement is straightforward . $\square$

Note that the first summand in (2.32) corresponds to the dynamics of Aldous model, and the second term considers the possibility of a site to being able to get open even neighboring an already frozen cluster.

Having at hand Lemma 10 the definition of a joint process on the directed edges of $\widehat{\mathbb{T}}$ follows in the same way as in [6]. Note that defining the process on directed edges instead of vertices is required to incorporate the information of the direction in which freezings occur. Let $\vec{E}$ denote the set of directed edges of $\mathbb{T}$. The random variables $\widehat{Y}_{\vec{e}}$ with $\vec{e} = \langle u, v \rangle \in \vec{E}$ should be thought as the time at which the vertex v gets frozen on the infinite tree rooted at v which is isomorphic to $\widehat{\mathbb{T}}$ and that does note contain the edge (u, v). For each vertex $u \in V$ we define the set of neighbor vertices of u

$$\partial u := \{v \in V : v \sim u\}. \tag{2.36}$$

Lemma 11. *(Lemma 4.1 in [19], Brouwer 2005) There exists a joint law for*
$\{(\widehat{Y}_{\langle u,v \rangle}, \tau_u)_{u \in \partial u}\}_{u \in V}$ *which is invariant under automorphisms of the 3-regular tree and such that for each $u \in V$ and each $v \in \partial u$ we have*

(i) For $\vec{e} \in \vec{E}$, $\widehat{Y}_{\vec{e}}$ has distribution $\widehat{\nu}$.

(ii) For each $\vec{e} = \langle u, v, \rangle$,

$$\widehat{Y}_{\vec{e}} = \varphi(\min\{\widehat{Y}_{\vec{e}_1}, \widehat{Y}_{\vec{e}_2}\}, \max\{\widehat{Y}_{\vec{e}_1}, \widehat{Y}_{\vec{e}_2}\}, \tau_u), \tag{2.37}$$

a.s., where $\vec{e}_1$ and $\vec{e}_2$ are the children of $\vec{e}$.

(iii) For each finite connected subset $S \subset V$, the variables $(\widehat{Y}_{\langle u,v \rangle} : u \in S, v \notin S)$ are independent of each other and independent of the collection $(\tau_u : u \in S)$.

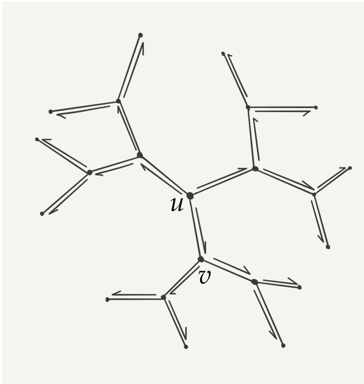

Figure 2.2: Directed tree

To define the modified frozen percolation process define for all $u \in V$,

$$Z_u := \min\{\widehat{Y}_{\langle v,u \rangle} : v \in \partial u\} \\ = \min\{\widehat{Y}_{\langle u,v \rangle} : v \in \partial u, Y_{\langle u,v \rangle} \geq \tau_u\}. \tag{2.38}$$

Note that the modified frozen percolation process cannot be defined as in Aldous process by first looking at the non frozen edges at time 1, in the modified version of the process it can happen that clusters that were active for a positive fraction of time still get frozen. For all $t \in [0,1]$

$$\mathcal{W}_t := \{v \in V : \tau_v > t\} \\ \mathcal{G}_t := \{v \in V : \tau_v \leq t, Z_v > t\} \\ \mathcal{R}_t := \{v \in V : \tau_v \leq t, Y_v \leq t\}. \tag{2.39}$$

Brouwer proved that the sets $\mathcal{W}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$, $t \in [0,1]$ indeed fit the initial description of the model in (2.26):

Lemma 12. *(Lemma 4.4 in [19], Brouwer 2005) Let $S_s(u)$ denote the cluster of u in $\mathcal{G}_s$, considered as a set of sites. Almost surely, $\forall u \in V$ with $Z_u < \infty$*

(i) $S_{Z_u^-(u)} \subseteq \mathcal{R}_{Z_u}$

(ii) $|S_{Z_u^-}(u)| = \infty$.

Lemma 13. *(Lemma 4.5 in [19], Brouwer 2005) Almost surely there is no infinite component in $\mathcal{G}_t$ for $t \in [0,1)$.*

We denote the law of the modified frozen percolation process as $\mathbf{P}_\infty$.

2.4 Non-existence in $\mathbb{Z}^2$

In order to illustrate the fact that the above construction of the frozen percolation process is not a trivial matter, we briefly describe the argument that shows that such a frozen percolation process cannot be defined in $\mathbb{Z}^2$ ([10] and [14]), or any other planar lattice where Russo-Seymour-Welsh type estimates hold.

Let us assume that a frozen percolation process exists and show that this leads to a contradiction: First, recall that for all $p \leq p_c = 1/2$, there is no infinite cluster for bond percolation on $\mathbb{Z}^2$, [39]. It follows that up to (and including) time $1/2$, the frozen percolation process is identical to the usual increasing coupling of Bernoulli percolation, where edges appear independently at constant rate.

Russo-Seymour-Welsh theory shows that at time $1/2$, almost surely, each point will be surrounded by infinitely many disjoint nested clusters. Let us for instance consider the set $\{A_i\}_{i\geq 1}$ of nested clusters (ordered from inwards to outwards) that do surround the origin.

Now, when $t > 1/2$, usual (non-frozen) percolation possesses infinite clusters, so this implies that at such a time, some infinite frozen clusters must exist in the frozen percolation process (if no clusters were frozen, then the frozen percolation process would be identical to the percolation process for $t' \in (1/2, t)$, and possess infinite clusters). On the other hand, all the edges that are open at time $1/2$ will remain opened at all times $t > 1/2$ in the frozen percolation process. Since any infinite connected set of $\mathbb{Z}^2$ has to intersect infinitely many A_i's, it follows that for all $t_0 > 1/2$, there exists almost surely one frozen infinite cluster $C(t_0)$, and that this cluster does contain all the A_i's for $i \geq i_0$. But the complement of this cluster can contain only finite components, because of the nesting of the A_i's. In particular, this shows that for all $t \geq t_0$, $C(t_0)$ will remain the only infinite cluster in the frozen percolation process. By monotonicity, we conclude that for all $t > 1/2$, $C(t) = \cap_{t_0 > 1/2} C(t_0)$ almost surely.

On the other hand, $C(t_0)$ is necessarily contained in the unique infinite (non-frozen) percolation cluster at time t_0 (because an edge that is opened for the frozen process is necessarily open for the ordinary percolation process). Hence, for all $t > 1/2$, $C(t)$ is a subset of the intersection of all infinite clusters for $t > 1/2$, which happens to be empty (because there is no infinite cluster at $p = 1/2$). So, we indeed reach a contradiction.

Note that this argument makes extensive use of planarity and is therefore not applicable for $\mathbb{Z}^d$ for $d > 2$.

Chapter 3

Distribution of finite clusters in the solution

3.1 On the d-regular tree

Aldous described the geometry of clusters in the frozen percolation process. A complete picture is given through Proposition 11 and Proposition 14 in [6]. After time $t_c(d)$ finite clusters are distributed as critical clusters in Bernoulli percolation and infinite clusters have the law of the infinite incipient cluster in percolation.
Knowing the distribution (obtained in the last chapter) of the time at which the root edge e_0 gets frozen on $\overline{\mathbb{T}}_d$, it is possible to obtain the law of connected components in the process. We revise this result in this section.

The independence of the random variables $\{\tau_e\}_{e \in E_d}$ allows to write in the model of Bernoulli percolation the distribution of the open cluster containing e_0 in $\mathcal{B}(\overline{\mathbb{T}}_d)$ at time t as,

$$P(B_t = \mathbf{T}) = P(e \in \mathcal{B}_t)^{|\mathbf{T}|} P(e \notin \mathcal{B}_t)^{|\partial_b(\mathbf{T})|}, \tag{3.1}$$

for $\mathbf{T}$ a finite subtree of $\overline{\mathbb{T}}_d$ containing e_0. Recall that $\partial_b \mathbf{T}$ denotes the

external edge boundary of $\mathbf{T}$.
The state of two different edges in the process of frozen percolation are clear to not being independent, but still by conditioning first in the edges on $\partial_b \mathbf{T}$ to not be in the process, the state of the edges in $\mathbf{T}$ is independent, we will use in repeatedly occasions this argument to obtain similarly results as the one in (3.1) for the various frozen percolation processes to treat, in particular we have the following lemma.

Lemma 14. *For all $t \in [0,1]$ and for all $\mathbf{T}$ finite subtree of $\overline{\mathbb{T}}_d$ containing e_0,*

$$\mathbb{P}_\infty \left(\mathcal{C}_t = \mathbf{T}\right) = t^{|\mathbf{T}|} \mathbb{P}_\infty \left(e_0 \notin \mathcal{A}_t\right)^{|\partial_b \mathbf{T}|}. \tag{3.2}$$

Proof. We fix $t \in [0,1]$. Let $\mathbf{T}$ be a finite subtree of $\overline{\mathbb{T}}_d$ containing e_0. For each $e \in \partial_b \mathbf{T}$ we denote the tree isomorphic to $\overline{\mathbb{T}}_d$ with edge root e that is not intersecting the tree $\mathbf{T}$ as T_e. We define $\mathcal{A}(e) = \{\mathcal{A}_t(e)\}_{t\in[0,1]}$ the frozen percolation process on T_e. We define the event

$$A_t(\mathbf{T}) := \bigcap_{e \in \partial_b \mathbf{T}} \{e \notin \mathcal{A}_t(e)\}. \tag{3.3}$$

Note that there is a natural coupling between $\mathcal{A}$ and $\{\mathcal{A}(e)\}_{e\in\partial_b \mathbf{T}}$. Consider an edge $e \in \partial_b \mathbf{T}$, $e = (u, v)$. Without loss of generality assume $u \in V(\mathbf{T})$ and $v \in V \setminus V(\mathbf{T})$. On the event $\{\mathcal{C}_t = \mathbf{T}\}$, the edge e is not in $\mathcal{A}_t$ if and only if e is not open in the frozen percolation on $\mathcal{A}(e)$ at time t. Indeed, if $e \notin \mathcal{A}_t(e)$ it is clear, e is not in $\mathcal{A}_t$. The case $e \in \mathcal{A}_t(e)$ but not in $\mathcal{A}_t$ can only happens if at some time in the interval $[0, t]$ the cluster containing the vertex $u \in V(\mathbf{T})$ belongs to an infinite cluster, this leads to two possibilities: either the edge in $\mathbf{T}$ containing u is not open at any time (because it was neighboring an infinite cluster at the moment that its clock rang) or it is in an infinite component, both options are not compatible with the event $\{\mathcal{C}_t = \mathbf{T}\}$ since $\mathbf{T}$ is finite, therefore we conclude that e

most be in $\mathcal{A}_t$. Hence,

$$\{\mathcal{C}_t = \mathbf{T}\} \cap \Big\{ \bigcap_{e \in \partial_b \mathbf{T}} \{e \notin \mathcal{A}_t\} \Big\} = \{\mathcal{C}_t = \mathbf{T}, A_t(\mathbf{T})\}. \tag{3.4}$$

On the other hand conditioning on the event $A_t(\mathbf{T})$ every edge of $\mathbf{T}$, during the time interval $[0, t]$, can only neighbour finite open connected components on the frozen percolation process. This implies that conditioning on the event $A_t(\mathbf{T})$ the state of edges on $\mathbf{T}$ are independent and the event $\{\mathcal{C}_t = \mathbf{T}\}$ occurs if and only if the clocks of edges on $\mathbf{T}$ has rang already by time t, that is

$$\mathbb{P}_\infty(\mathcal{C}_t = \mathbf{T} | A_t(\mathbf{T})) = t^{|\mathbf{T}|}. \tag{3.5}$$

Since the processes $\{\mathcal{A}(e)\}_{e \in \partial_e \mathbf{T}}$ are defined on disjoint sets of edges and the trees T_e are all isomorphic to $\overline{\mathbb{T}}_d$, the process $\{\mathcal{A}(e)\}_{e \in \partial_b \mathbf{T}}$ are independent processes all having the same law as $\mathcal{A}$. Therefore the probability of $A_t(\mathbf{T})$ is given by

$$\mathbb{P}_\infty \left(\bigcap_{e \in \partial_b \mathbf{T}} \{e \notin \mathcal{A}_t(e)\} \right) = \mathbb{P}_\infty \left(e_0 \notin \mathcal{A}_t\right)^{|\partial_b \mathbf{T}|}. \tag{3.6}$$

By the last two equations in (3.5) and (3.6), using (3.4) it holds

$$\begin{aligned} \mathbb{P}_\infty \left(\mathcal{C}_t = \mathbf{T}\right) &= \mathbb{P}_\infty \left(\mathcal{C}_t = \mathbf{T}, \bigcap_{e \in \partial_b \mathbf{T}} \{e \notin \mathcal{A}_t\} \right) \\ &= \mathbb{P}_\infty \left(\mathcal{C}_t = \mathbf{T}, A_t(\mathbf{T})\right) \\ &= t^{|\mathbf{T}|} \mathbb{P}_\infty \left(e_0 \notin \mathcal{A}_t\right)^{|\partial_b \mathbf{T}|}, \end{aligned} \tag{3.7}$$

for all $t \in [0, 1]$. □

In the following lemma we use the explicit distribution of the random variables Y to get the explicit probability that the edge e_0 is not open in the frozen percolation process on $\overline{\mathbb{T}}_d$ at time t. We define for $t \in [0,1]$,

$$\gamma_\infty(t) := \mathbb{P}_\infty(e_0 \notin \mathcal{A}_t). \tag{3.8}$$

Before going to the lemma we need to recall from Chapter 1 that the radius of convergence of the d-Catalan generating function C_d was denoted in the first chapter as R_d and it was proved to be

$$R_d = \frac{1}{d-1}\left(\frac{d-1}{d-2}\right)^{d-2}. \tag{3.9}$$

Lemma 15.

$$\gamma_\infty(t) = \begin{cases} 1-t & \textit{for } 0 \leq t \leq \frac{1}{d-1} \\ \\ (R_d/t)^{1/(d-2)} & \textit{for } \frac{1}{d-1} \leq t \leq 1. \end{cases} \tag{3.10}$$

Since this probability plays an important role in computing the law of clusters as we saw in the last lemma, we present the details of the proof of Lemma 15. Similar computations are done by Aldous in Proposition 2 in [6].

Proof. There are two possibilities for the edge e_0 to not being in the set $\mathcal{A}_t$, either its clock has not rang yet by time t or the clock rang at some time smaller than t and just before its clock rang the vertex v_0 it was already in an infinite connected component of $\overline{\mathbb{T}}_d \setminus \{e_0\}$, written in terms of the random variables $\{Y_{\vec{e}}\}_{e \in \vec{E}_d}$

$$\tau_{e_0} > \min\{Y_{\vec{e}_1}, \cdots, Y_{\vec{e}_{d-1}}\}, \tag{3.11}$$

where $\vec{e}_1, \ldots, \vec{e}_{d-1}$ are the incident edges to e_0 directed away from it. Thus

$$\mathbb{P}_\infty(e_0 \notin \mathcal{A}_t) = \mathbb{P}_\infty(\tau_{e_0} \leq t) + \mathbb{P}_\infty\left(\tau_{e_0} \leq t, \min\{Y_{\vec{e}_1}, \cdots, Y_{\vec{e}_{d-1}}\} < \tau_{e_0}\right). \tag{3.12}$$

The right summand in the last line is equal to 0 for $t \leq t_c(d)$ since no infinite clusters exists before this time a.s. This proves the lemma

for times in $[0, t_c(d)]$.
For times $t \in [t_c(d), 1]$, conditioning with τ_{e_0} and integrating over all possibles values for τ_{e_0}, the probability in (3.12) is equal to

$$1 - t + \int_{t_c(d)}^{t} (1 - \mathbb{P}_\infty(s \leq \min\{Y_{\vec{e}_1}, \cdots, Y_{\vec{e}_{d-1}}\}))ds, \tag{3.13}$$

from which we obtain by the independence of $Y_{\vec{e}_1}, \ldots, Y_{\vec{e}_{d-1}}$ that

$$\mathbb{P}_\infty(e_0 \notin \mathcal{A}_t) = 1 - t_c(d) - \int_{t_c(d)}^{t} (1 - F(s))^{d-1} \, ds. \tag{3.14}$$

Plugging in (2.14) in to the right hand side of the last equation and using that $t_c(d) = (d-1)^{-1}$, we get

$$\mathbb{P}_\infty(e_0 \notin \mathcal{A}_t) = \frac{d-2}{d-1} - \frac{1}{(d-1)^{(d-1)/(d-2)}} \int_{t_c(d)}^{t} s^{-(d-1)/(d-2)} ds \tag{3.15}$$

which is equal to

$$\frac{d-2}{d-1} + \frac{d-2}{(d-1)^{(d-1)/(d-2)}} \left(\frac{1}{t^{1/(d-2)}} - (d-1)^{1/(d-2)} \right), \tag{3.16}$$

and that can be rewritten as

$$\frac{d-2}{(d-1)^{(d-1)/(d-2)}} t^{-1/(d-2)} = (R_d/t)^{1/(d-2)}.$$

This concludes the proof for $t \in [t_c(d), 1]$. □

We prove next Proposition 11 in [6] in the general case for d-regular trees. The prove that we give here differs from the proof in [6] of Proposition 11, we write probabilities in terms of the function γ_∞, this suits more the strategies to be followed in the coming sections treating with the convergence to Aldous' model.

Proposition 16. *(Proposition 11 in [6], Aldous 2011) At all times in* $[t_c(d), 1]$ *the cluster of* e_0 *in the frozen percolation process on* $\overline{\mathbb{T}}_d$ *conditioned to be finite and non empty, has the same law as the Bernoulli open* *percolation cluster of e_0 on $\overline{\mathbb{T}}_d$ conditioned to be non empty.*

Proof. Let $\mathbf{T}$ be a finite subtree of $\overline{\mathbb{T}}_d$ containing e_0. By Lemma 14 and Lemma 15, for $t \in [t_c(d), 1]$

(3.17)
$$\mathbb{P}_\infty\left(\mathcal{C}_t = \mathbf{T}\right) = t^{|\mathbf{T}|}\left(\frac{R_d}{t}\right)^{|\partial_b \mathbf{T}|/(d-2)} = R_d^{|\mathbf{T}|}\left(\frac{R_d}{t}\right)^{1/(d-2)},$$

where we have used that $|\partial_b \mathbf{T}| = |\mathbf{T}|(d-2)+1$. Note that this probability depends on $\mathbf{T}$ just only through $|\mathbf{T}|$. Since the cardinality of the set

$$\{\mathbf{T} \text{ subtrees of } \overline{\mathbb{T}}_d \text{ containing } e_0 \text{ with } |\mathbf{T}| = k\}, \tag{3.18}$$

is given by the number c_k defined as in (1.21), by (3.17) it holds that for $t \geq t_c(d)$,

$$\begin{aligned} \mathbb{P}_\infty(0 < |\,\mathcal{C}_t\,| < \infty) &= \sum_{k=1}^{\infty} \mathbb{P}_\infty(|\,\mathcal{C}_t\,| = k) \\ &= \left(\frac{R_d}{t}\right)^{1/(d-2)} \sum_{k=1}^{\infty} c_k R_d^k. \end{aligned} \tag{3.19}$$

The last line is equal to

$$\left(\frac{R_d}{t}\right)^{1/(d-2)} \left(C_d(R_d) - 1\right). \tag{3.20}$$

Using that $\{\mathcal{C}_t = \mathbf{T}\} \subseteq \{0 < |\,\mathcal{C}_t\,| < \infty\}$ for all $\mathbf{T}$ finite, by the last two displays and (3.17)

$$\mathbb{P}_\infty\left(\mathcal{C}_t = \mathbf{T} | 0 < |\,\mathcal{C}_t\,| < \infty\right) = \frac{R_d^{|\mathbf{T}|}}{(C(R_d) - 1)}. \tag{3.21}$$

By Lemma 5, the above ratio is equal to

$$(d-2)\left(\frac{(d-2)^{d-2}}{(d-1)^{d-1}}\right)^{|\mathbf{T}|}, \tag{3.22}$$

According to (1.11) the quantity in the last display is equal to the probability that the cluster of e_0 in Bernoulli percolation at time $t_c(d)$ is equal to $\mathbf{T}$ given that e_0 is open at time $t_c(d)$. □

3.2 On Galton-Watson trees

We now see what happens to the shape of clusters in the solution for Galton-Watson trees. Consider the frozen percolation process on a Galton-Watson tree with offspring distribution $\pi = (\pi_j)_{j\geq 0}$. Again, we are focusing here only on the case where $t > t_c$, since in the subcritical case, the freezing has no effect.

Here, there are different ways to interpret what the information that a cluster in the Galton-Watson tree means:

(i) This could mean that this information contains the number of (closed or frozen) outgoing edges of each vertex in the cluster, and possibly the actual position of these closed/frozen edges among the (ordered) descendent outgoing edges at this vertex (mind that of course, these positions are distributed uniformly among all possible choices).

(ii) Or this could mean that the only information about the cluster is provided by its sites and the open vertices that join them. For instance, the knowledge of how many closed outgoing edges a leaf of the tree has is not provided.

For instance, for general offspring distributions, the information of the number of outgoing edges of the root in the initial Galton-Watson tree is not provided by the information of the cluster containing the root. In the case of the d-regular tree, there was no essential difference between these two interpretations, because the number of closed/frozen outgoing edges could be inferred from the cluster.

Let us first look at the first interpretation. Then, Lemma 14 can be generalized as follows: When $\mathbf{T}$ is a possible realization of the information provided by the cluster containing e_0 (so this is a tree with the information for each of its sites x, of its denote by its number $J(x)$ of offsprings, including the closed/frozen edges that are not in the tree). We denote by $|\partial_b \mathbf{T}|$ its total number of outgoing closed or frozen edges. Then

$$\mathbb{P}_\infty\left(\mathcal{C}_t = \mathbf{T}\right) = \left(\prod_x \pi_{J(x)}\right) \times t^{|\mathbf{T}|} \times \mathbb{P}_\infty\left(e_0 \notin \mathcal{A}_t\right)^{|\partial_b \mathbf{T}|}, \tag{3.23}$$

and it would be possible to make the analysis starting from there.

However, from now on, we choose to focus on the second interpretation of what the information about the cluster that contains the origin means. Our first remark is the following:

Lemma 17. *The law of $\mathcal{C}_t$ conditioned to be finite is that of a Galton-Watson tree with some offspring distribution $(\kappa_j(t))_{j\geq 0}$, conditioned to be finite.*

Proof. Let $I(x)$ denote the number of descendents of x in the tree $\mathcal{C}_t$. We only have to see that for each $t > t_c$ and each finite tree $\mathbf{T}$, the probability that $\mathcal{C}_t = \mathbf{T}$ is a multiple of $\prod_{x\in\mathbf{T}}(\kappa_{I(x)}(t))$ for some $(\kappa_j(t))_{j\geq 0}$.

Let us define

$$\Gamma(t) := \mathbb{P}_\infty\left(e_0 \notin \mathcal{A}_t\right)$$

(mind that this event can occur either because the clock of that edge did not ring, or because it could not open because of the freezing of one of its adjacent edges). In other words, when $t > t_c$, $\Gamma(t)$ is given by a formula of the type

$$\Gamma(t) = (1-t) + \sum_{j\geq 1} j\pi_j \int_0^t (1-A(s))^{j-1}(t-s)dA(s).$$

Then, we can estimate the probability that $\mathcal{C}_t = \mathbf{T}$ by summing over all possible values $J(x) \geq I(x)$ that represent the number of actual offsprings of x in the Galton-Watson tree, and one gets indeed the desired formula for

$$\kappa_i(t) := \sum_{j \geq i} \pi_j \frac{j!}{i!(j-i)!} t^i \Gamma(t)^{j-i}.$$

□

In general, the expression for $\kappa_i(t)$ turns out to be messy, but there are nevertheless some offspring distributions $(\pi_j)_{j\geq 0}$ for which things are nice:

- The first example is the case of the geometric off-spring distribution (given by $\pi_j = (1-u)u^j$ when $j \geq 0$). Recall that in this case, the probability to observe a given finite tree as the Galton-Watson tree is proportional to $(1-u)^s \times u^e$ where s is the number of sites of the tree and u its number of edges. But since $s = e+1$, we get that the probability is proportional to $(1-u)(u(1-u))^e$.

  Let us perform the whole summation leading to the above expression of $\kappa_i(t)$ again in that case: Suppose that we are given a finite tree **T** rooted at the origin. We let $I(x)$ denote the number of off-springs of a vertex x that do lie in (the vertex x is therefore a leaf of the tree if and only if $I(x) = 0$).

  We can then compute the probability that at time t, the cluster that contains the root edge e_0 in our frozen percolation process is equal to exactly the tree . For this, we only have to sum the contributions of all possible configurations of the number of descendents (in the original Galton-Watson tree) of all vertices of **T**.

  For instance, for a vertex x with $I(x)$ descendents in , we require that:

  - This vertex had at least $I(x)$ descendents in the original Galton-Watson tree.

  - Among its $j \geq I(x)$ descendent edges, exactly $I(x)$ are open, and the other $j - I(x)$ are closed.

This leads to a contribution of the type (here we sum over all j and all positions of the $I(x)$ open edges among the $j = i_1 + 1 + i_2 + 1 + \ldots + 1 + i_{I(x)+1}$ descendents).

$$\sum_{i_1,\ldots,i_{I(x)+1}\geq 0} u^{i_1}B(t)^{i_1} \times (ut) \times u^{i_2}B(t)^{i_2} \times (ut) \times \ldots \times u^{i_{I(x)+1}}B(t)^{i_{I(x)+1}} \times (1-u)$$

for the site x. This sum can be rewritten as

$$(1-u)\frac{(ut)^{I(x)}}{(1-uB(t))^{I(x)+1}} = \frac{1-u}{1-uB(t)} \times \left(\frac{ut}{1-uB(t)}\right)^{I(x)}.$$

Note that for the leaves y of , where $I(y) = 0$, the same formula actually holds.

Hence, the probability that $\mathcal{C}_t$ is equal to a tree $\mathbf{T}$ with s vertices and e edges is proportional to

$$\left(\frac{1-u}{1-uB(t)}\right)^{s} \times \left(\frac{ut}{1-uB(t)}\right)^{e}.$$

Using again that the total number s of sites of the tree is equal to one plus the total number e of edges of the tree (each edge points to its descendent site), we get that the probability that the cluster containing the origin at time t is exactly is equal to a constant multiple (that depends on t) of

$$\left(\frac{u(1-u)t}{(1-uB(t))^2}\right)^{e}.$$

Interestingly, up to the multiplicative factor, this formula is that of a Galton-Watson tree with some geometric off-spring distribution of the type $\kappa_i(t) = u(t)^i(1-u(t))$, conditioned to be finite.

- Suppose that we are in the case where $\pi_j = 0$ for all $j > 2$, so that the initial Galton-Watson tree consists of vertices of degree 2 or 3 (in the molecular description, only atoms of Oxygen and Natrium). Then, each vertex in the finite tree $\mathcal{C}_t$ can also have only 0, 1 or 2 descendents, so that $\kappa_j(t) = 0$ for all $j > 2$ as well. But if one samples such a Galton-Watson tree $\mathbf{T}$ and looks at the tree $\widehat{\mathbf{T}}$ consisting only of its "interior vertices" (the vertices in $\widehat{\mathbf{T}}$ are the vertices of $\mathbf{T}$ with exactly two descendents in $\mathbf{T}$), then this new tree $\widehat{\mathbf{T}}$ (conditioned to be non-empty) is also a Galton-Watson tree with some offspring distribution $(\widehat{\kappa}_j)$ with $\widehat{\kappa}_j = 0$ for all $j > 2$ and for $j = 1$. Similarly, for all $t > t_c$, one has a distribution $(\widehat{\kappa}_j(t))_{j\geq 0}$ with these properties.

We now notice the following fact that we already mentioned in the introduction:

Proposition 18. *When $t > t_c$, then the offspring distribution $(\kappa_j(t))_{j\geq 0}$ is critical in the sense that $\sum_j j\kappa_j(t) = 1$.*

Sketch. Suppose that $t > t_c$ and that $(\kappa_j(t))_{j\geq 0}$ is not critical. Then, it is well-known that the law of a Galton-Watson tree with this offspring distribution conditioned to be finite is that of a subcritical Galton-Watson tree (where the expected number of offsprings is strictly smaller than 1), for a reference see Proposition 5.23 in [45]. In particular, the expected number of sites for this subcritical Galton-Watson tree is finite. Similarly, the law π of the number of neighbors of the subcritical tree within the original Galton-Watson tree has an exponential decay, and its expectation is finite. But then, if we apply the coalescence rules to these subcritical Galton-Watson trees, then for some positive time, no infinite cluster will appear. Indeed, this evolution can be upper-bounded by the percolation process with small parameter on the Galton-Watson tree with offspring distribution π. In particular, this would imply that the freezing probability is constant on $[t, t+s]$ for some positive s. But this contradicts the fact that this probability is increasing in $[t_c, 1]$. $\square$

The previous two observations have therefore nice consequences:

- In the first case (geometric offspring distributions), one must have $u(t) = 1/2$ for all $t > t_c$.

- In the second case ($\pi_j = 0$ for all $j > 2$, one must have $\widehat{\kappa}_0(t) = \widehat{\kappa}_2(t) = 1/2$ for all $t > t_c$.

This type of observation will be important later in this thesis. Indeed, the same heuristics can be applied to the case of finite clusters in Brouwer's frozen site-percolation model on the 3-regular tree, so that one expects stationarity of the law of the subtree of *internal vertices* of non-frozen trees in the solution for that model. We will see this fact popping up in our analysis of this model, and it will instrumental in the proof of our finite-freezing approximation result.

Note that in the generic case, the law of $\mathcal{C}_t$ conditioned to be finite is not going to be constant with respect to t. One can for instance look at the second case above (where $\pi_j = 0$ for all $j > 2$) and see that $\kappa_1(t)$ will not necessarily be constant.

3.3 For the site-percolation model

We now come back to the case of the frozen site percolation model on binary trees introduced and defined by Brouwer. As we saw in the more general setting of Galton-Watson trees the law of finite clusters in the frozen percolation after time t_c is not necessarily invariant and the same holds for the modified version of frozen percolation. Brouwer observed that the distribution of finite clusters is not invariant in the modified frozen percolation process (see (5.1) in [19]), the distribution always depends on time during the whole interval $[0, 1]$.

Using the combinatorial structure of subtrees of $\mathbb{T}$, we show that some invariance in the modified process also holds after the critical time $1/2$. One way to understand this is through ϕ, the transformation on trees introduced in the Preliminaries chapter.

Proposition 19. *For all $t \in [1/2, 1]$ the law of $\phi(\mathcal{C}_t)$ the tree of internal vertices of $\mathcal{C}_t$ conditioning the root vertex to be an internal vertex is distributed as a Bernoulli critical cluster conditioned to be non empty.*

In order to express the law of non-frozen clusters we define the function for $t \in [0,1]$

$$\widehat{\gamma}_\infty(t) := \mathbf{P}_\infty(v_0 \in \mathcal{R}_t). \tag{3.24}$$

From (2.29) and Lemma 10 it holds that,

$$\widehat{\gamma}_\infty(t) = \begin{cases} 0 & \text{for } 0 \le t \le 1/2 \\ \ln(2t) & \text{for } 1/2 \le t \le 1. \end{cases} \tag{3.25}$$

To prove Proposition 19 we need to recall some facts on the combinatorics of $\mathbb{T}$ and introduce some functions first.

Consider $\mathbf{T}$ a subtree of $\widehat{\mathbb{T}}$ containing the root vertex v_0. The vertices of $\mathbf{T}$ can be classified in to three types: internal vertices (with no neighbors in $V \setminus V(\mathbf{T})$), type I vertices (with one neighbor in $V \setminus V(\mathbf{T})$) and type II vertices (with two neighbors in the set $V \setminus V(\mathbf{T})$). We recall the notation introduced in Section 1.3.1, $\mathcal{I}(\mathbf{T})$ denotes the set of internal vertices of $\mathbf{T}$, and $\partial_1\mathbf{T}$ denotes the set of vertices of type I. Their respective sizes are denoted by $I(\mathbf{T})$ and $J(\mathbf{T})$.

We can write the law of the open cluster containing v_0 at time t in terms of two probabilities related to the the type of v_0 in $\mathcal{C}_t$. We consider the probabilities $\mathbf{P}_\infty\left(V(\mathcal{C}_t) = v_0\right)$, the probability that $\mathcal{C}_t$ is the trivial tree, and $\mathbf{P}_\infty(v_0 \text{ is of type I}|v_1 \text{ or } v_2 \in \mathcal{G}_t)$ the conditional probability that v_0 is of type I on $\mathcal{C}_t$ given that v_1 or v_2 is in $\mathcal{G}_t$. We define the functions on $[0,1]$,

$$x_\infty(t) := \mathbf{P}_\infty\left(V(\mathcal{C}_t) = v_0\right) \tag{3.26}$$

and

$$y_\infty(t) := \mathbf{P}_\infty\left(v_0 \text{ is of type I}|\; v_1 \text{ or } v_2 \in \mathcal{G}_t\right) \tag{3.27}$$

for $t \in (0,1]$ and $y_\infty(0) = 0$.

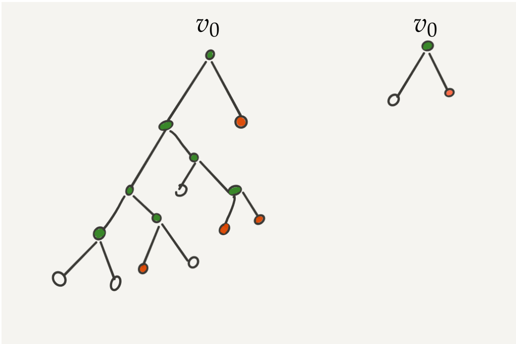

Figure 3.1: On the right one configuration where $\mathcal{C}_t$ is the trivial tree and on the left a configuration where v_0 is of type I

The following lemma is the analogous version of Lemma 14.

Lemma 20. *For all $t \in [0,1]$, and for all $\mathbf{T}$ a finite subtree of $\widehat{\mathbb{T}}$ containing v_0, it holds that*

(i)

$$\mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}) = t^{I(\mathbf{T})} x_\infty(t)^{I(\mathbf{T})+1} y_\infty(t)^{J(\mathbf{T})}. \tag{3.28}$$

(ii)

$$x_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s))^2 ds, \text{ and } y_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s)) ds. \tag{3.29}$$

Proof. The second part of the statement of the lemma follows by noting that for the event $\{V(\mathcal{C}_t) = v_0\}$ to hold it is required that $\tau_{v_0} \leq t$, that v_1 and v_2 are not in $V(\mathcal{C}_t)$ and that v_0 has not gotten frozen until time t. The last two requirements can occur in two ways, because the clocks of v_1 and v_2 did not ring until time t or because they were frozen already by the time τ_{v_0}. Moreover before time τ_{v_0} the states of v_1 and v_2 are independent and their states are the states in the frozen percolation processes defined on $\widehat{T}^1$ and $\widehat{T}^2$ respectively. Hence,

$$x_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s))^2 ds \tag{3.30}$$

Similarly for y_∞, conditioning on the event that v_1 or v_2 are open at time t, the vertex v_0 is of type I if and only if exactly one of its neighbors is not open at time t, this can occur as before only if the neighbor vertex was already frozen by time τ_{v_0} or if his clock has not rang by time t, therefore we see that

$$y_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s)) ds. \tag{3.31}$$

To prove the first part of the lemma, denote the external boundary of $\mathbf{T}$ by $\widetilde{\partial}\mathbf{T} := \{v \in V \setminus V(\mathbf{T}) : v \sim u \text{ for some } u \in \partial\mathbf{T}\}$. For every $u \in \widetilde{\partial}\mathbf{T}$ we denote by $\mathbf{T}_u$ the binary tree isomorphic to $\mathbf{T}$ which is rooted at u and which its intersection with $\mathbf{T}$ is empty. We denote by $V_u := V(\mathbf{T}_u)$ its set of vertices. We note that for all $v \in \partial\mathbf{T}$ and $u \in \widetilde{\partial}\mathbf{T}$ with $u \sim v$, on the event $\{\mathcal{C}_t = \mathbf{T}\} \cap \{\tau_v \le t\}$ the event $\{u \in \mathcal{R}_{\tau_v}\}$ depends uniquely on the collection $\{\tau_v\}_{v \in V_u}$ and on the modified frozen percolation on the tree $\mathbf{T}_u$, that is on the event $\{\mathcal{C}_t = \mathbf{T}\} \cap \{\tau_v \le t\}$, the event $\{u \in \mathcal{R}_{\tau_v}\}$ occur if and only if $\{u \in \mathcal{R}^{\mathbf{T}_u}_{\tau_v}\}$, where $\mathcal{R}^{\mathbf{T}_u}_{\tau_v}$ is the set of frozen vertices at time τ_v on the modified percolation process defined on $\mathbf{T}_u$. Therefore, it holds that

$$\begin{aligned}&\mathbf{P}_\infty\left(\{\mathcal{C}_t = \mathbf{T}\} \cap \{\bigcap_{v \in \partial\mathbf{T}} \{\tau_v \le t\}\} \cap \{\bigcap_{v \in \partial\mathbf{T}} \bigcap_{u \sim v} \{u \in \mathcal{R}_{\tau_v}\} \cup \{u \in \mathcal{W}_t\}\}\right)\\ &= \mathbf{P}_\infty\left(\{\mathcal{C}_t = \mathbf{T}\} \cap A_t(\mathbf{T})\right),\end{aligned} \tag{3.32}$$

with

$$A_t(\mathbf{T}) := \{\bigcap_{v \in \partial\mathbf{T}} \{\tau_v \le t\}\} \cap \{\bigcap_{v \in \partial\mathbf{T}} \bigcap_{u \sim v} \{u \in \mathcal{R}^{\mathbf{T}_u}_{\tau_v}\} \cup \{u \in \mathcal{W}_t\}\}. \tag{3.33}$$

Since $|\mathbf{T}| < \infty$, conditioning on the event $A_t(\mathbf{T})$, the event $\{\mathcal{C}_t =$

$\mathbf{T}\}$ occurs if and only if the clocks of all internal vertices of $\mathbf{T}$ have already rang by time t. That is,

$$\mathbf{P}_\infty\left(\mathcal{C}_t = \mathbf{T} | A_t(\mathbf{T})\right) = t^{I(\mathbf{T})}. \tag{3.34}$$

Moreover $\{\{\tau_v \leq t\} \bigcap \{\{u \in \mathcal{R}_{\tau_v}^{\mathbf{T}_u}\} \cup \{u \in \mathcal{W}_t\}\}\}_{v\in\partial\mathbf{T}}$ is a collection of independent events. Conditioning on $\{\tau_v\}_{v\in\partial\mathbf{T}}$ and integrating over the times at which the clocks associated to vertices in $\partial\mathbf{T}$ ring and with $\mathbf{s} = (s_v)_{v\in\partial\mathbf{T}} \in [0,1]^{|\partial\mathbf{T}|}$,

$$\mathbf{P}_\infty(A_t(\mathbf{T})) = \prod_{v\in\partial\mathbf{T}} \int_{[0,t]^{|\partial C|}} \mathbf{P}_\infty\left(\bigcap_{u\sim v} \{u \in \mathcal{R}_{s_v}^{\mathbf{T}_u}\} \cup \{u \in \mathcal{W}_t^{\mathbf{T}_u}\}\right) d\mathbf{s}. \tag{3.35}$$

By the transitivity of $\mathbb{T}$, for all $0 < s < t \leq 1$, the event $\{u \in \mathcal{R}_s^{\mathbf{T}_u}\} \cup \{u \in \mathcal{W}_t^{\mathbf{T}_u}\}$ has the same probability as the event $\{v_0 \in \mathcal{R}_s\} \cup \{v_0 \in \mathcal{W}_t\}$. So the equation in the last display is equal to

$$\begin{aligned} &\prod_{v\in\partial C} \int_{[0,t]^{|\partial C|}} \prod_{u\sim v} (1 - t + \widehat{\gamma}_\infty(s)) d\mathbf{s} \\ &= \left(\int_{[0,t]} (1 - t + \widehat{\gamma}_\infty(s))^2 ds\right)^{I(\mathbf{T})+1} \left(\int_{[0,t]} (1 - t + \widehat{\gamma}_\infty(s)) ds\right)^{J(\mathbf{T})}. \end{aligned} \tag{3.36}$$

On the other hand it holds that $\{\mathcal{C}_t = \mathbf{T}\} \subset A_t(\mathbf{T})$, therefore (3.34), (3.35) and (3.36) together imply that

$$\begin{aligned} &\mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}) \\ &= t^{I(\mathbf{T})} \left(\int_{[0,t]} (1 - t + \widehat{\gamma}_\infty(s))^2 ds\right)^{I(\mathbf{T})+1} \left(\int_{[0,t]} (1 - t + \widehat{\gamma}_\infty(s)) ds\right)^{J(\mathbf{T})}. \end{aligned} \tag{3.37}$$

□

Having this lemma at hand we prove the invariance for the law of $\phi(\mathcal{C}_t)$ on $[1/2, 1]$.

Proof of Proposition 19. We start by computing the probability that v_0 is an internal vertex of $\mathcal{C}_t$. This happens if and only if $\tau_{v_0}, \tau_{v_1}, \tau_{v_2} \leq t$ and v_1 and v_2 have not been frozen until time t in their respective processes on $\widehat{T}_{v_1}$ and $\widehat{T}_{v_2}$, that is

$$\mathbf{P}_\infty(v_0 \in \mathcal{I}(\mathcal{C}_t)) = t(t - \widehat{\gamma}_\infty(t))^2 = x_\infty(t), \tag{3.38}$$

where the las equality holds just by computing the integral in (ii) in Lemma 20.

Now we compute the probability $\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, v_0 \in \mathcal{I}(\mathcal{C}_t)\right)$. We start accounting for all possible values of the number of vertices of type I of $\mathcal{C}_t$.

$$\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, v_0 \in \mathcal{I}(\mathcal{C}_t)\right) = \sum_{j=0}^{\infty} \mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, J(\mathcal{C}_t) = j\right). \tag{3.39}$$

Thanks to Lemma 20 we know that for all trees $\mathbf{T}$ with $I(\mathbf{T}) = i$ and $J(\mathbf{T}) = j$

$$\mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}) = t^i x_\infty(t)^{i+1} y_\infty(t)^j, \tag{3.40}$$

therefore, for all $j \geq 0$ the probaility $\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, J(\mathcal{C}_t) = j\right)$ is equal to $t^i x_\infty(t)^{i+1} y_\infty(t)^j$ times the number of trees containing v_0 with v_0 being an internal vertex that has tree of internal vertices equal to $\widehat{\mathbf{T}}$. Similarly as explained in Subection 1.3.2 this number is $\binom{2i+j-1}{j} 2^j$. This yields for all $t \in [0, 1]$,

$$\begin{aligned} &\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, v_0 \in \mathcal{I}(\mathcal{C}_t), J(\mathcal{C}_t) = j\right) \\ &= \binom{2i+j-1}{j} t^i x_\infty^{i+1}(t)(2y_\infty(t))^j. \end{aligned} \tag{3.41}$$

Therefore from (3.39) and (3.41)

$$\begin{aligned}
&\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \widehat{\mathbf{T}}, v_0 \in \mathcal{I}(\mathcal{C}_t)\right) \\
&= x_\infty(t)\,(tx_\infty(t))^i \sum_{j=0}^{\infty} \binom{2i+j-1}{j} (2y_\infty(t))^j \\
&= x_\infty(t) \frac{\left(tx_{\infty(t)}\right)^i}{(1-2y_\infty(t))^{2i}}.
\end{aligned} \tag{3.42}$$

By the last equation and (3.38) we get that,

$$\mathbf{P}_\infty\left(\phi(\mathcal{C}_t) = \mathbf{T} | v_0 \in I(\mathcal{C}_t)\right) = \left(\frac{tx_\infty(t)}{(1-2y_\infty(t))^2}\right)^i. \tag{3.43}$$

By using the integral expressions in (ii) in Lemma 20 for $x_\infty(t)$ and $y_\infty(t)$, after some computations we obtain that for all $t \in [1/2, 1]$

$$\frac{tx_\infty(t)}{(1-2y_\infty(t))^2} = \frac{1}{4} \tag{3.44}$$

which concludes the proof.

(3.45)

□

In a very similar way we obtain further information on the geometry of the active cluster of v_0 at time t.

Lemma 21. *Conditioning on the event $\{I(\mathcal{C}_t) = i, v_0 \in \mathcal{G}_t\}$ then J it is distributed as the sum of $2i+1$ geometric random variables with parameter $2y_\infty(t)$.*

Proof. Recall that

$$a_{ij} = |\{\mathbf{T} \text{ subtree of } \hat{\mathbb{T}} \text{ containing } v_0 : I(\mathbf{T}) = i, J(\mathbf{T}) = j\}|. \tag{3.46}$$

Summing over the possibilities of the number of vertices of type I of $\mathcal{C}_t$ and using Lemma 20

$$\mathbf{P}_\infty\left(I(\mathcal{C}_t) = i, v_0 \in \mathcal{G}_t\right) = \sum_{j=0}^{\infty} a_{i,j} t^i x_\infty^{i+1} y_\infty^j. \tag{3.47}$$

Using the expression (1.69) for a_{ij} in the above line we get

$$\begin{aligned}\mathbf{P}_\infty\left(I(\mathcal{C}_t) = i, v_0 \in \mathcal{G}_t\right) &= c_i t^i x_\infty^{i+1} \sum_{j=0}^{\infty} \binom{2i+j}{j} (2y_\infty)^j \\ &= c_i x_\infty \frac{(tx_\infty)^{i+1}}{(1-2y_\infty)^{2i+1}}.\end{aligned} \tag{3.48}$$

Then we have that

$$\mathbf{P}_\infty(J(\mathcal{C}_t) = j, I(\mathcal{C}_t) = i, v_0 \in \mathcal{G}_t) = c_i x_\infty (tx_\infty)^i \binom{2i+j}{i} (2y_\infty)^j. \tag{3.49}$$

By the last two equations (3.49) and (3.50):

$$\mathbf{P}_\infty(J(\mathcal{C}_t) = j | I(\mathcal{C}_t) = i) = \binom{2i+j}{i} (2y_\infty)^j (1-2y_\infty)^{2i+1}, \tag{3.50}$$

which is the distribution of a negative binomial random variable with parameters $2i+1$ and $2y_\infty(t)$. (Recall that the sum of r independent geometric distributions with parameters p is distributed as a negative binomial distribution with parameters r and p). $\square$

Through these two lemmas we have a whole description of finite clusters. Before time $1/2$ clusters start to grow just as in Bernoulli percolation on sites during the the time interval $[0, 1/2]$. After time $1/2$ the tree of internal vertices is critical at all times, and conditioning on the number of internal vertices the number of vertices of type I is a negative binomial random variable, in particular

$$\mathbf{E}_\infty[J(\mathcal{C}_t) | I(\mathcal{C}_t)] = \frac{2y_\infty(t)(2I(\mathcal{C}_t)+1)}{1-2y_\infty(t)}. \tag{3.51}$$

Note that the function $2y_\infty(t)/(1-2y_\infty(t))$ is an increasing function on $t \in [0,1/2]$ taking the value 1 at time $1/2$ and strictly decreasing $[1/2,1]$.

3.4 Comment on the infinite frozen clusters

In the present chapter, we have focused on the law of the finite (non-frozen) clusters in the frozen percolation process. In his paper [6], Aldous did also describe the law of the infinite frozen clusters, and he showed that they were (each) distributed like the incipient infinite percolation cluster (IIC) on the binary tree. In particular, for the percolation model on the 3-regular tree, all frozen clusters have the same randon distribution, and this distribution is independent of the time at which they did freeze. Recall also that the IIC can be viewed as the law of the cluster of the origin for $p = 1/2$ percolation on the binary tree "conditioned to be infinite" (i.e., the appropriately taken limit at $n \to \infty$ of the law of the cluster, conditioned to be of size at least n), so that it can be viewed as a Galton-Watson tree conditioned to be infinite.

Let us now quickly and heuristically explain what part of this statement one could generalize to the other models that we have discussed in this chapter. Let us focus on the case of frozen percolation on the Galton-Watson trees. Recall that we argued that in that case, when $t > t_c$, the law of the finite trees in the solution is that of a critical Galton-Watson tree with distribution $\kappa(t) = (\kappa_j(t))_{t\geq 0}$. Then:

- The law of an infinite cluster that is born (i.e., that becomes infinite) exactly at time t will be that of a Galton-Watson tree for the distribution $\kappa(t)$, conditioned to be infinite (in the same sense as before).

- So, in the binary tree and in the geometric offspring case where the law $\kappa(t)$ does not depend on $t > t_c$, the shapes of all individual frozen clusters will be the same.

- In the general case where $\kappa(t)$ is not constant with respect to

t, this means that when observing an infinite frozen cluster, it will be possible to recover its freezing time.

Note that one way to approach and understand this type of features is to study the mechanisms of freezing large finite clusters, that will be the topic of the last two chapters of this thesis.

Chapter 4

Freezing of large clusters of the d-regular tree

4.1 On d-regular trees

The N-parameter model of frozen percolation was first defined on the binary tree in [12] by van den Berg, Kiss and Nolin. Instead of freezing infinite clusters as Aldous did, clusters are frozen when their sizes reach the threshold size $N \in \mathbb{N}$.

We will consider in this section the N-parameter frozen percolation model on the slightly more general setting of d-regular trees. Recall that $\overline{\mathbb{T}}_d = (\overline{V}_d, \overline{E}_d)$ denotes the planted d-regular tree, v_* is the only vertex with degree one, e_0 is the edge adjacent to v_*, and v_0 is the other extremity of e_0. As seen before, frozen percolation on the (not planted) d-regular tree $\mathbb{T}_d$ can be well described once we understand frozen percolation on the planted tree. Hence, in this chapter, we work only with frozen percolation on $\overline{\mathbb{T}}_d$. We fix $d \geq 3$. Let $\{\tau_e\}_{e \in \overline{E}_d}$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. At time $t = \tau_e$ the edge e gets to be open if the open clusters of its two end vertices have size smaller than N. In this process, the size of open clusters is therefore always smaller than $2N - 1$.

The existence of the finite parameter version of frozen percolation on d-regular trees is provided by standard arguments in the theory of interacting particle systems. Indeed, after the change of time $\kappa_e := -\ln(1-\tau_e)$ for $e \in \overline{E}_d$ (so that κ_e has an exponential distribution), the N-parameter frozen percolation on $\overline{\mathbb{T}}_d$ can be viewed as a Markovian finite-range interacting particle system. Classical arguments on the existence of such processes can be found in [44].

We denote for all $N \in \mathbb{N}$ the law of the frozen percolation process with parameter N as $\mathbb{P}_N$. Analogously as for the infinite parameter model, we write $\mathcal{A}_t$ for the set of *activated edges*, i.e. the edges of $\overline{E}_d$ that are open at time t in the frozen percolation process. Then we define the functions $\gamma_N(t)$ (depending on d, but as we did before we omit this in the notation). We define on the time interval $[0,1]$ the function $\gamma_N(t) : [0,1] \to [0,1]$ as

$$\gamma_N(t) := \mathbb{P}_N(e_0 \notin \mathcal{A}_t). \tag{4.1}$$

By the same arguments as in the proof of Lemma 14 we can write the distribution of the cluster $\mathcal{C}_t$ containing v_* on the N- parameter frozen percolation process in terms of γ_N.

Lemma 22. *For all $N \in \mathbb{N}$, for all $\mathbf{T}$ a finite subtree of $\overline{\mathbb{T}}_d$ containing v_* with $|\mathbf{T}| < N$, and for all $t \in [0,1]$,*

$$\mathbb{P}_N\left(\mathcal{C}_t = \mathbf{T}\right) = t^{|\mathbf{T}|}\gamma_N(t)^{|\partial_b(\mathbf{T})|} = t^{|\mathbf{T}|}\gamma_N(t)^{1+(d-2)|\mathbf{T}|}. \tag{4.2}$$

We recall that $|\mathbf{T}|$ above denotes the number of edges in $\mathbf{T}$. Let us now present the first part of the proof of Theorem 1 in [12] in the general setting of d-ary trees:

Proposition 23. *For all $\mathbf{T}$ finite subtree of $\overline{\mathbb{T}}_d$ containing v_* and for all $t \in [0,1]$,*

$$\mathbb{P}_N(\mathcal{C}_t = \mathbf{T}) \to \mathbb{P}_\infty(\mathcal{C}_t = \mathbf{T}), \tag{4.3}$$

uniformly on $[0,1]$ as $N \to \infty$.

We now explain the strategy to follow in the proof of the proposition. From Lemmas 14 and 22 we know that for all times $t \in [0,1]$,

$$\mathbb{P}_N\left(\mathcal{C}_t = \mathbf{T}\right) = \gamma_N(t)\left(t\gamma_N(t)^{(d-2)}\right)^{|\mathbf{T}|} \tag{4.4}$$

and

$$\mathbb{P}_\infty\left(\mathcal{C}_t = \mathbf{T}\right) = \gamma_\infty(t)\left(t\gamma_\infty(t)^{(d-2)}\right)^{|\mathbf{T}|} \tag{4.5}$$

using that $|\partial_b \mathbf{T}| = |\mathbf{T}|(d-2)+1$ for all finite subtrees of $\overline{\mathbb{T}}_d$ containing v_*. From (4.4) and (4.5) we see that it is enough, in order to prove Proposition 23 to prove that γ_N converges uniformly to γ_∞ on $[0,1]$. We prove this in two steps. First, in Lemma 24 we prove the convergence on $t \le t_c(d)$, which follows easily from the absence of infinite cluster for Bernoulli percolation (without freezing) on this interval. Then we prove the convergence for $t \ge t_c(d)$ by using that γ_N satisfies a simple differential equation.

Lemma 24. *Uniformly on* $[0, t_c(d)]$,

$$\gamma_N \underset{N\to\infty}{\longrightarrow} \gamma_\infty. \tag{4.6}$$

Proof. We use the coupling between the N-parameter frozen percolation and Bernoulli percolation on $\overline{\mathbb{T}}_d$. For all $t \in [0, t_c(d)]$,

$$\mathbb{P}_N(\tau_{e_0} \le t, e_0 \notin \mathcal{A}_t) = \int_0^t \mathbb{P}_N(|\,\mathcal{C}_s^*\,| \ge N-1)ds \tag{4.7}$$

where $\mathcal{C}_s^*$ denotes the open cluster containing v_0 in the frozen percolation process on $\overline{\mathbb{T}}_d \setminus \{e_0\}$. Since any edge open at time t on the N-parameter frozen percolation is open in the Bernoulli percolation at time t, the right hand side in the above equation is bounded by

$$t_c(d)P(|B^*_{t_c(d)}| \ge N-1), \tag{4.8}$$

with $B^*_{t_c(d)}$ denoting the open cluster containing v_0 on the Bernoulli percolation process on $\overline{\mathbb{T}}_d \setminus \{e_0\}$ at time $t_c(d)$. The above probability goes to 0 as $N \to \infty$. This implies the desired uniform convergence. □

In order to prove the convergence on $t \geq t_c(d)$, we define for all $N \in \mathbb{N} \cup \{\infty\}$ the functions

$$\psi_N(t) := t\gamma_N(t)^{d-2}, \tag{4.9}$$

for $t \in [0,1]$. We will prove convergence for ψ_N instead of proving convergence of γ_N directly. To prove the convergence of ψ_N to ψ_∞ we will show that ψ_N satisfies a differential equation.

Recall the partial sums associated to the d-Catalan numbers, $C_N(z) = \sum_{k=0}^{N} c_k z^k$ for $z \in \mathbb{R}$, and define for $N \in \mathbb{N}$ the function F_N

$$F_N(t,\psi) := \frac{\psi}{t}(d-2)\left(C(R_d) - C_N(\psi)\right) \tag{4.10}$$

and its infinite parameter analogous function F_∞,

$$F_\infty(t,\psi) := \frac{\psi}{t}(d-2)\left(C(R_d) - C(\psi)\right), \tag{4.11}$$

where $C(z) = \sum_{k=0}^{\infty} c_k z^k$ as in (1.24) and recall that the power series C has radius of convergence R_d (see (1.26)) and it is finite on the interval $[0, R_d]$.
Note that $F_N(t, \psi_N(t))$ is well defined for all $t \in [0,1]$ and for all $N \in \mathbb{N} \cup \{\infty\}$.

Lemma 25. *For all $N \in \mathbb{N} \cup \{\infty\}$, the function $\gamma_N(t)$ is differentiable and satisfies*

$$\psi_N'(t) = F_N(t, \psi_N(t)) \tag{4.12}$$

for $t \in [0,1]$.

Proof. We start noting that e_0 joins the process at time $s = \tau_{e_0}$ if and only if the cluster of v_0 on the N-parameter frozen percolation on

$\overline{\mathbb{T}}_d \setminus \{e_0\}$ has size less than N. Therefore by conditioning on τ_{e_0},

$$\mathbb{P}_N(e_0 \notin \mathcal{A}_t) = 1 - \int_0^t \mathbb{P}_N(|\,\mathcal{C}_s^*\,| \le N-1)ds, \tag{4.13}$$

where $\mathcal{C}_s^*$ denotes the cluster containing the vertex v_0 on the process on $\overline{\mathbb{T}}_d \setminus \{e_0\}$.
Now we note that the number of subtrees of $\overline{\mathbb{T}}_d$ containing the vertex v_0 and not containing e_0 with k edges is given by c_{k+1}, by very similar arguments to the ones used to prove Lemma 22,

$$\mathbb{P}_N(|\,\mathcal{C}_s^*\,| = k) = c_{k+1} s^k \gamma_N(s)^{d-1+k(d-2)} \tag{4.14}$$

because the probability of the event $\{\mathcal{C}_t = \mathbf{T}\}$ only depends on $\mathbf{T}$ through $|\mathbf{T}|$. Then we can rewrite the right side of (4.13) as

$$1 - \int_0^t \gamma_N(s)^{d-1} \sum_{k=0}^{N-1} c_{k+1} \left(s\gamma_N(s)^{d-2}\right)^k ds, \tag{4.15}$$

which is equal to

$$1 - \int_0^t \frac{\gamma_N(s)}{s} \sum_{k=1}^{N} c_k (s\gamma_N(s)^{d-2})^k ds. \tag{4.16}$$

Therefore, from the above displays we get that

$$\gamma_N(t) = 1 - \int_0^t \frac{\gamma_N(s)}{s} (C_N(s\gamma_N(s)^{d-2}) - 1) ds. \tag{4.17}$$

Notice that $|\gamma_N(u) - \gamma_N(v)| \le \mathbb{P}(\tau_{e_0} \in [u,v]) \le |u-v|$ for every $0 \le u,v \le 1$. Hence γ_N is continuous. Then, the equation above implies that γ_N is differentiable and for every t,

$$\gamma_N'(t) = -\frac{\gamma_N(t)}{t} (C_N(t\gamma_N(t)^{d-2}) - 1). \tag{4.18}$$

It is immediate from the differentiability of γ_N that ψ_N is differentiable as well and that

$$\psi_N'(t) = \gamma_N(t)^{d-2} + (d-2)t\gamma_N(t)^{d-3}\gamma_N'(t). \tag{4.19}$$

Making the change of variables in (4.9) and using that $C(R_d) = (d-1)/(d-2)$, see Lemma 5, we obtain that

$$\psi_N'(t) = \frac{\psi_N(t)}{t}(d-2)\left(C(R_d) - C_N(\psi_N(t))\right). \tag{4.20}$$

□

Having Lemma 25 at hand one could feel that the proof of Lemma 24 is finished. The function C has radius of convergence R_d. If it happens that uniformly on t and N the numbers ψ_N are well inside of the disc of convergence of C (bounded away from R_d), C is analytic at these points and the the partial sums $C_N(\psi_N)$ should converge uniformly to $C(\psi_\infty)$, so we have uniform convergence of F_N to F_∞ (actually we would have convergence of all derivatives of γ_N) and then by Lemma 25 the convergence of ψ_N to ψ_∞ would follow. But actually after the time $t_c(d)$, $\psi_N(t)$ lies outside of the region of convergence of C!

In order to give more insight on the situation let us recall the Bernoulli percolation process on $\overline{\mathbb{T}}_d$. The law of the number of edges $|B_t|$ in the open cluster B_t of v_* in Bernoulli percolation is given by

$$P(|B_t| = k) = c_k\left(t(1-t)^{d-2}\right)^k(1-t) \tag{4.21}$$

(at this point the reader could intuit already from the presence of the generating functions C_N and C in F_N and F_∞ that these functions are related to the size of open clusters in frozen percolation). The function $t \mapsto t(1-t)^{d-2}$ has a maximum at $t_c(d)$, and at this point the function takes its maximal value R_d. Thus we see that for times bounded away from t_c the tails of the size of finite clusters decay exponentially fast. It is just at the critical time that the size of open clusters have heavy tails.
As we see thanks to the coupling of the processes $\mathcal{A}$ and $\mathcal{B}$ for times

bounded away from above by $t_c(d)$ we are in the situation of percolation and the clusters in $\mathcal{A}$ are subcritical clusters, for this times the convergence can be proved as we said before using that F_N converges to F_∞ uniformly. For times after $t_c(d)$ the change of the dynamics from Bernoulli percolation to frozen percolation causes that after time $t_c(d)$ the system is somehow in a mixture of the critical and supercritical regimes, there exists infinite clusters but the small clusters (non-frozen) have a heavy tail, they are critical. If we expect convergence of the finite-parameter version we should expect the same behaviour for non-frozen clusters after time $t_c(d)$. For that reason we need to proceed differently for $t \in [t_c(d), 1]$. Although we would not be able to prove directly convergence of $F_N(t, \psi_N(t))$ to $F_\infty(t, \psi_{\infty(t)})$ the situation is not that bad, we will prove that after $t_c(d)$, ψ_N is outside from the region of convergence indeed, but as N goes to infinite ψ_N approaches the boundary sufficiently fast so that the convergence of ψ_N to ψ_∞ holds. More concretely, we will use Lemma 25 to get differential inequalities that will imply bounds on ψ_N that warranties convergence.

We would like to note that until this point the proof of Proposition 23 follows the same ideas used in [12]. The next proof of Lemma 26 is where the difference of our proof and the one in [12] lies. Our proof uses classical methods in ODE's to get bounds on functions by proving differential inequalities. Instead of trying to use the differential equation in (4.20) to get an implicit equation for γ_N we will use it to obtain differential inequalities that will imply Lemma 26. It is this approach that allows us to carry on the proof to models with more complicated freezing rules, where trying to get an implicit equation for γ_N seems more challenging.

By Lemma 15 we have $\psi_\infty(t) = R_d$ for every $t \geq t_c(d)$. Furthermore, recall that for $N \geq 1$ the number w_N (defined at the end of Section 1.2) is the unique positive root of $C_N(z) - C(R_d) = 0$ and we have seen that $w_N \downarrow R_d$ as N tends to infinity. Therefore, the uniform convergence of γ_N to γ_∞ on $[0, 1]$ (hence, the proof of Proposition 23) follows from Lemma 24 together with the following lemma.

Lemma 26. *For every N large enough, it holds that for all $t \in [t_c(d), 1]$,*

$$R_d \leq \psi_N(t) < w_N. \tag{4.22}$$

Proof. Set $b_N := \psi_N(t_c(d))$. First we prove that $\psi_N(t) < w_N$ for every $t \geq t_c(d)$. By Lemma 24, b_N converges to $R_d < w_N$, therefore $b_N < w_N$ for N large enough. Furthermore we observe that for all $N \in \mathbb{N}$ and for all $t \in [t_c(d), 1]$,

$$F_N(t, w_N) = \frac{w_N}{t}(d-2)\left(C(R_d) - C_N(w_N)\right) = 0 \tag{4.23}$$

by the definition of w_N. Thus the constant function w_N is an upper fence for any solution of the I.V.P.:

$$\begin{cases} \psi'(t) = F_N(t, \psi(t)), & t \in [t_c(d), 1]. \\ \psi(t_c(d)) = b_N, \end{cases} \tag{4.24}$$

in particular for ψ_N thanks to Lemma 25. Therefore we can conclude that $\psi_N < w_N$ on $[t_c(d), 1]$.

Now we prove the lower bound in (4.22). On the interval $[0, R_d]$, where the generating function C is defined it holds that $C_N < C$, in particular $C_N(R_d) < C_\infty(R_d)$ and this implies that for every $t \geq t_c(d)$

$$F_N(t, R_d) = \frac{R_d}{t}(d-2)\left(C(R_d) - C_N(R_d)\right) > 0. \tag{4.25}$$

This condition and the equation above ensures that R_d is a lower fence for $\gamma_N(t)$ on $[t_c(d), 1]$ due to Lemma 25 and because $b_N > R_d$ (which follows from $\gamma_N(t) \geq 1 - t$ applied to $t = t_c(d)$). Therefore, $\psi_N \geq R_d$ on $[t_c(d), 1]$. $\square$

4.2 Good size functions

In [12] the previous result of convergence towards Aldous's model was proved to hold in more generality. Instead of freezing according to the number of edges, freezing occurs when the size of clusters given by a good size function exceeds a finite parameter $N \in \mathbb{N}$. We see how the strategy used in the last section still works, mainly the one used to prove Lemma 26. We present the results on the generality of d-regular trees.
Recall the definition of a good size function:

Definition 27. *Let $\mathcal{T}_d$ be the subset of finite connected subgraphs of $\mathbb{T}_d$. A* good size function *$s : \mathcal{T}_d \to \mathbb{N}$, is a function satisfying the following conditions:*

- *(i) Compatibility with homomorphisms. For all $\mathbf{T} \in \mathcal{T}_d$ a finite subgraph of the d-regular tree and injective homomorphisms h we have $s(h(\mathbf{T})) = s(\mathbf{T})$.*
- *(ii) Finiteness. For all $N \in \mathbb{N}$ and for every vertex v, the set $\{\mathbf{T} \in \mathcal{T}_d | v \in V(\mathbf{T}), s(\mathbf{T}) \leq N\}$ is finite.*
- *(iii) Monotonicity. If $\mathbf{T}, \mathbf{T}' \in \mathcal{T}_d$ with $\mathbf{T} \subseteq \mathbf{T}'$, then $s(\mathbf{T}) \leq s(\mathbf{T}')$.*
- *(iv) Boundedness above by the volume. For all $\mathbf{T} \in \mathcal{T}_d$, we have $s(\mathbf{T}) \leq |\mathbf{T}|$, with $|\mathbf{T}|$ the number of edges of $\mathbf{T}$.*

The finite parameter model of frozen percolation with good size function s is proved to exists by the same arguments that for the particular case of s being the volume graph, (iv) warranties that the model can be seen as a finite range interacting particle system. We denote by $\mathbb{P}_N^{(s)}$ the law of the finite parameter model with good size function s.

It is proved in [12] that a change on the rules to freeze clusters (according to a good size function) does not brake the convergence to Aldous' model as N goes to ∞.

Proposition 28. *Let s be a good size function for the d-regular tree. For all $t \in [0,1]$, and all finite subtrees $\mathbf{T}$ of $\overline{\mathbb{T}}_d$,*

$$\mathbb{P}_N^{(s)}\left(\mathcal{C}_t = \mathbf{T}\right) \longrightarrow \mathbb{P}_\infty\left(\mathcal{C}_t = \mathbf{T}\right) \text{ as } N \to \infty. \tag{4.26}$$

In order to use the methods used in the above section we define for a good size function s the generating function

$$G_N^{(s)}(x) := \sum_{k=0}^{\infty} a_{k,N}^{(s)} x^k, \tag{4.27}$$

where $a_{k,l}$ is defined as

$$a_{k,l} := |\{\mathbf{T} \in \mathcal{T}_k : s(\mathbf{T}) \leq l-1\}|, \tag{4.28}$$

for $k, l \geq 1$.
We also define the sequence of roots $\{w_N^{(s)}\}_{N \geq 1}$. The function $xG_N^{(s)}(x)$ is an strictly increasing function for $x \in [0,\infty)$ taking value 0 at 0 and moreover $xG_N^{(s)}(x)$ tends to ∞ as $x \to \infty$, then we can define the numbers $w_N^{(s)}$ for all $N \geq 1$ as the unique positive root of the polynomial $xG_N^{(s)}(x) - \frac{1}{d-2} = 0$.

The strategy on the proof of Proposition 28 is the same as in the last section. The same arguments as in the proof of Lemma 14 lead to the lemma:

Lemma 29. *For all $t \in [0,1]$, and for all $\mathbf{T}$ a subtree of $\overline{\mathbb{T}}$ containing e_0,*

$$\mathbb{P}_N^{(s)}\left(\mathcal{C}_t = \mathbf{T}\right) = t^{|\mathbf{T}|}\left(\gamma_N^{(s)}(t)\right)^{|\partial_b \mathbf{T}|}. \tag{4.29}$$

In order to prove Proposition 28 we prove

Lemma 30. *$\gamma_N^{(s)}$ converges uniformly on $[0,1]$ to $\gamma_\infty^{(s)}$.*

In order to prove the lemma we actually prove convergence of

$$\psi_N^{(s)}(t) := t\gamma_N^{(s)}(t)^{d-2}. \tag{4.30}$$

for $t \in [0,1]$ to ψ_∞ as in (4.9). On the interval $[0, t_c(d)]$, the convergence follows from the absence of infinite cluster for the corresponding Bernoulli percolation process (as in Lemma 24). On the interval $[t_c(d), 1)$, the convergence follows from the fact that $\psi_N^{(s)}$ is a solution to the I.V.P.

$$\begin{cases} \psi'(t) = F_N^{(s)}(t, \psi(t)), & t \in [t_c(d), 1] \\ \phi(t_c(d)) = b_N^{(s)}, \end{cases} \tag{4.31}$$

with

$$F_N^{(s)}(t, \psi) := \frac{\psi}{t}\left(1 - \psi G_N^{(s)}(\psi)\right) \quad \text{and} \quad b_N^{(s)} := \psi_N^{(s)}(t_c(d)). \tag{4.32}$$

Proposition 31. *For all $N \in \mathbb{N}$, the function $\psi_N^{(s)}$ is differentiable and satisfies*

$$\psi_N^{(s)\prime}(t) = F_N^{(s)}(t, \psi_N^{(s)}(t)). \tag{4.33}$$

As the proof follows the same steps of the proof of Lemma 25 we save us some details this time.

Proof. For all $t \in [0,1]$,

$$\mathbb{P}_N^{(s)}(e_0 \notin \mathcal{A}_t) = 1 - \int_0^t \mathbb{P}_N^{(s)}\left(s(\mathcal{C}_u^*) \le N\right) du, \tag{4.34}$$

because of (ii) in Definition 27,

$$\int_0^t \mathbb{P}_N^{(s)}\left(s(\mathcal{C}_u^*) \le N\right) du = \int_0^t \sum_{k=0}^{\infty} \sum_{\substack{\mathbf{T} \in \mathcal{T}_k \\ s(\mathbf{T}) \le N}} \mathbb{P}_N^{(s)}\left(\mathcal{C}_s^* = \mathbf{T}\right) du. \tag{4.35}$$

We have seen in Lemma 29 that the probability $\mathbb{P}_N^{(s)}\left(\mathcal{C}_t = \mathbf{T}\right)$ depends on **T** only through $|\mathbf{T}|$, therefore we get

$$\text{(4.36)}\quad \mathbb{P}_N^{(s)}(e_0 \notin \mathcal{A}_t) = 1 - \int_0^t \sum_{k=0}^{\infty} a_{k,N}(u\gamma_N^{(s)}(u)^{d-2})^k \gamma_N^{(s)}(u)^{d-1} du.$$

Taking the derivative in the above equation yields to

$$\text{(4.37)}\qquad \gamma_N^{(s)\prime}(t) = -\gamma_N^{(s)}(t)^{d-1} G_N^{(s)}\left(t\gamma_N^{(s)}(t)^{d-2}\right).$$

this implies

$$\text{(4.38)}\qquad \psi_N^{(s)\prime}(t) = \frac{\psi_N^{(s)}(t)}{t}(d-2)\left(\frac{1}{d-2} - \psi_N^{(s)} G_N^{(s)}(\psi_N^{(s)})\right).$$

Note that the differentiability of $\gamma_N^{(s)}$ and $\psi_N^{(s)}$ follows from the same arguments as for the good size function $\mathbf{T} \mapsto |\cdot|$. □

We use the previous lemma to prove:

Lemma 32. *For all $t \in [t_c(d), 1]$,*

$$\text{(4.39)}\qquad R_d \leq \psi_N^{(s)}(t) \leq w_N^{(s)}.$$

Proof. We see that thanks to Proposition 31 that the constant functions R_d and $w_N^{(s)}$ are respectively lower and upper fences for the function $\psi_N^{(s)}$ on $[t_c(d), 1]$. First, using the convergence $\gamma_N^{(s)} \downarrow \gamma_\infty$ on $[0, t_c(d)]$, we have for every N large enough, $R_d < \psi_N^{(s)}(t_c(d)) < w_N^{(s)}$. Then, for all $t \in [t_c(d), 1]$, we have $F_N^{(s)}(t, w_N^{(s)}) = 0$. Therefore, Proposition 31 implies that $w_N^{(s)}$ is an upper fence for the function $\psi_N^{(s)}$ on $[t_c(d), 1]$.

To see that R_d is a lower fence for $\psi_N^{(s)}$ on the interval $[t_c(d), 1]$ we observe that

$$\text{(4.40)}\qquad F_N^{(s)}(t, R_d) \geq F_\infty(t, R_d) = 0$$

on the interval $[t_c(d), 1]$. $\square$

4.3 Freezing according to a random size

In this section, we illustrate the robustness of the approach presented in this chapter by explaining how to adapt it to the following framework: We consider a more general "randomized" freezing rule, and we explain the main ideas that allow to show the convergence also for that model.

Consider the planted tree $\overline{\mathbb{T}}_d = (\overline{V}_d, \overline{E}_d)$ of degree d. To every vertex $v \in \overline{V}_d$ associate independently a Bernoulli random variable X_v such that $\mathbb{P}(X_v = 0) = \mathbb{P}(X_v = 1) = 1/2$. Define the *$X$-size* of a set $S \subset \overline{V}_d$ by

$$X(S) := \sum_{v \in S} X_v. \tag{4.41}$$

Notice that the X-size of a fixed set S is a Binomial random variable with parameter $(|S|, 1/2)$. Now, we define a N-frozen percolation process according to the X-size. Let $\{\tau_e\}_{e \in \overline{E}_d}$ be a sequence of independent random variables that are uniformly distributed on $[0, 1]$, and that are independent of the X_v's. At time $t = \tau_e$ the edge e gets to be open (or activated) if the open clusters of its two end vertices have X-size smaller than N. It is again not a problem to see that this process is well-defined, because every finite set also has a finite X-size almost surely. We denote the law of this process by $\widetilde{\mathbb{P}}_N$. Let us now state the following convergence proposition, which is the analogue of Proposition 23 for this more general model.

Proposition 33. *For all $\mathbf{T}$ finite subtree of $\overline{\mathbb{T}}_d$ containing v_* and for all $t \in [0, 1]$,*

$$\widetilde{\mathbb{P}}_N(\mathcal{C}_t = \mathbf{T}) \to \mathbb{P}_\infty(\mathcal{C}_t = \mathbf{T}), \tag{4.42}$$

uniformly on $[0, 1]$ as $N \to \infty$.

We just sketch the main steps of the proof, since it is very similar to the proof in Section 4.1. That previous proof did involve a truncated version C_N of the generating function C for the d-Catalan numbers. Here, the proof will go along the very same lines, except that the function C_N is replaced by a different truncation function $\widetilde{C}_N$. The main difference is the part of the proof which explains how the function $\widetilde{C}_N$ is actually constructed, and that we now describe.

By the same arguments as in the proof of Lemma 14 we can write the distribution of the cluster $\mathcal{C}_t$ containing v_* in terms of

$$\widetilde{\gamma}_N(t) := 1 - \widetilde{\mathbb{P}}_N(e_0 \text{ is activated at time } t). \tag{4.43}$$

The edge e_0 is activated at time t if it rang at a time $s \leq t$ and the cluster $\mathcal{C}_s^*$ of v_0 in $\overline{\mathbb{T}} \setminus \{e_0\}$ at that time has X-size strictly smaller than N. Conditioning on the time τ_{e_0} at which the edge e_0 rings, we get

$$\begin{aligned} \widetilde{\gamma}_N(t) &= 1 - \widetilde{\mathbb{P}}_N(e_0 \text{ is activated at time } t) \\ &= 1 - \int_0^t \widetilde{\mathbb{P}}_N(X(\mathcal{C}_s^*) \leq N-1) ds. \end{aligned} \tag{4.44}$$

The probability in the integral can be easily computed by conditioning on the size of $\mathcal{C}_s^*$. For every s, we have

$$\widetilde{\mathbb{P}}_N(X(\mathcal{C}_s^*) \leq N-1) \tag{4.45}$$

$$= \sum_{i=0}^{\infty} \underbrace{\widetilde{\mathbb{P}}_N(X(\mathcal{C}_s^*) \leq N-1 \mid |\mathcal{C}_s^*| = i)}_{p_{i+1,N}} \widetilde{\mathbb{P}}_N(|\mathcal{C}_s^*| = i), \tag{4.46}$$

where $p_{i+1,N}$ is equal to the probability that a binomial random variable with parameters $(i, 1/2)$ is smaller than $N-1$ (for a large fixed N, one can think of $i \mapsto p_{i,N}$ as a "truncation" function behaving like $\mathbf{1}_{i \leq N/2}$). As in Eq. (4.14), we have

$$\widetilde{\mathbb{P}}_N(|\mathcal{C}_s^*| = i) = c_{i+1} s^i \widetilde{\gamma}_N(s)^{d-1+i(d-2)}, \tag{4.47}$$

and the three displayed equations above lead to the the functional equation

$$\widetilde{\gamma}_N(t) = 1 - \int_0^t \frac{\gamma_N(s)}{s}(\widetilde{C}_N(s\widetilde{\gamma}_N(s)^{d-2}) - 1)ds, \tag{4.48}$$

where we defined

$$\widetilde{C}_N(r) = \sum_{i=0}^{\infty} c_i p_{i,N} r^i, \tag{4.49}$$

with the convention $p_{0,N} = 1$. Eq. (4.48) above is the same as Eq. (4.17) with γ_N and C_N replaced by $\widetilde{\gamma}_N$ and $\widetilde{C}_N$ respectively. The rest of the proof is the same as in Section 4.1: defining $\widetilde{w}_N$ by $\widetilde{C}_N(\widetilde{w}_N) = C(R_d)$, we can use a "fence" method to prove that for every $t \in [t_c(d), 1]$

$$R_d \leq t\gamma_N^{d-2} \leq \widetilde{w}_N, \tag{4.50}$$

which does conclude the proof of the proposition.

Chapter 5

Freezing of large clusters for the site-percolation model

5.1 Notation and reminders

We first recall the definitions and the main results for the frozen-site percolation, and then define the N-parameter model (which is a truncated version of the model). In this chapter, we are working with the tree $\widehat{\mathbb{T}}$, defined in Section 1.3: it the infinite tree where every vertex has degree 3, except the root v_0 which has degree 2. The law of the frozen-site percolation on $\widehat{\mathbb{T}}$ (as defined in Section 2.3) is denoted by $\mathbf{P}_\infty$. Recall that $\mathcal{W}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$ denote the sets of white (the vertices whose clocks have not rung at time t), green (the open vertices at time t), and red vertices (the frozen vertices at time t) at time t respectively. As seen in Chapter 3, the process is well understood through the function

$$\widehat{\gamma}_\infty(t) := \mathbf{P}_\infty(v_0 \in \mathcal{R}_t), \tag{5.1}$$

which is equal to

$$\widehat{\gamma}_\infty(t) = \begin{cases} 0 & \text{for } 0 \le t \le 1/2 \\ \ln(2t) & \text{for } 1/2 \le t \le 1. \end{cases} \tag{5.2}$$

In Lemma 20, we proved that the law of the cluster $\mathcal{C}_t$ of v_0 at time t is determined by

$$\mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}) = t^{I(\mathbf{T})} x_\infty(t)^{I(\mathbf{T})+1} y_\infty(t)^{J(\mathbf{T})}, \tag{5.3}$$

for all finite subtree $\mathbf{T}$ of $\widehat{\mathbb{T}}$ containing v_0, where

$$x_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s))^2 ds, \text{and } y_\infty(t) = \int_0^t (1 - t + \widehat{\gamma}_\infty(s)) ds. \tag{5.4}$$

The two functions x_∞ and y_∞ are not constant in the "supercritical" regime $t \ge 1/2$ (and therefore the law of $\mathcal{C}_t$ conditioned to be finite is not invariant when t varies, contrary to the case of bond-frozen percolation). It will handier to analyse the truncated versions of the model via the following function than x_∞ and y_∞:

$$r_\infty(t) := \frac{t x_\infty(t)}{(1 - 2y_\infty(t))^2}. \tag{5.5}$$

This function can be interpreted as the parameter governing the law of the tree $\phi(\mathcal{C}_t)$ of internal vertices of $\mathcal{C}_t$. More precisely, in the proof of Proposition 19, we showed that the tree $\phi(\mathcal{C}_t)$ is a Bernoulli percolation tree and that it has size i with a probability that is proportional to $r_\infty(t)^i$ (see Eq. (3.43)). In (3.44), we established that

$$\forall t \ge 1/2 \quad r_\infty(t) = \frac{1}{4}, \tag{5.6}$$

and in Proposition 19, we proved that the law of the tree $\phi(\mathcal{C}_t)$ is always a critical percolation tree when $t \ge 1/2$. Our startegy will be to study the parameter r_N governing the law of the tree of internal vertices and, using the methods of the previous chapter, we will prove its convergence to $r_\infty(t) = 1/4$.

5.2 Freezing of large clusters for the site-percolation model

For all $N \in \mathbb{N}$, we define the N-parameter model of modified frozen percolation as the natural cut-off version of the model in [19]: As soon as a green cluster reaches a size greater than N, it immediately freezes and all vertices in that cluster change their state from open or active to frozen. Recall that some of the main ingredients of the proof of [12], where convergence of the cut-off model (for edge-percolation) to Aldous' model was established, do not hold true anymore for this site-percolation freezing model, so that some new ideas are required.

Existence in the model (for each N) follows again by standard results on the theory of interacting particle systems. We denote the law of the N-parameter modified frozen percolation as $\mathbf{P}_N$. As before, we write $\mathcal{W}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$ for the sets of white (the vertices whose clocks have not rung at time t), green (open), and red (frozen) vertices at time t respectively. We introduce analogous functions to the ones introduced for the infinite parameter modified frozen percolation process. For all $N \in \mathbb{N}$ and $t \in [0,1]$,

$$\widehat{\gamma}_N(t) := \mathbf{P}_N(v_0 \in \mathcal{R}_t), \tag{5.7}$$

and the probabilities

$$x_N(t) := \mathbf{P}_N\left(V(\mathcal{C}_t) = v_0\right) \tag{5.8}$$

and

$$y_N(t) := \mathbf{P}_N\left(v_0 \text{ is of type I } | v_1 \text{ or } v_2 \in \mathcal{G}_t\right). \tag{5.9}$$

As in Lemma 20 we can express the law of the open cluster containing v_0 at time t in terms of the functions $x_N(t)$ and $y_N(t)$.

Lemma 34. *For all $N \in \mathbb{N}$, for all $t \in [0,1]$, and for all $\mathbf{T}$ a finite subtree* *of $\widehat{\mathbb{T}}$ containing v_0 with $|\mathbf{T}| < N$, it holds that*

$$\mathbf{P}_N(\mathcal{C}_t = \mathbf{T}) = t^{I(\mathbf{T})} x_N^{I(\mathbf{T})+1} y_N^{J(\mathbf{T})},$$

and

$$x_N(t) = \int_0^t (1 - t + \widehat{\gamma}_N(s))^2 ds, \qquad y_N(t) = \int_0^t (1 - t + \widehat{\gamma}_N(s)) ds.$$

We omit the proof as it is identical as the one of Lemma 20 – one just has to replace ∞ by N.

The goal of this chapter is to present the proof of the following convergence result:

Theorem 35. *For all $\mathbf{T}$ finite subtrees of $\widehat{\mathbb{T}}$ containing v_0, and for all $t \in [0,1]$,*

$$\mathbf{P}_N(\mathcal{C}_t = \mathbf{T}) \longrightarrow \mathbf{P}_\infty(\mathcal{C}_t = \mathbf{T}), \text{ as } N \to \infty. \tag{5.10}$$

5.3 Warm-up

Combined with Lemma 20 and Lemma 34, we see that the following lemma does imply Theorem 35.

Lemma 36. *The functions x_N and y_N converge uniformly to x_∞ and y_∞ respectively on $[0,1]$.*

The remainder of this chapter will therefore be devoted to the proof of the lemma. The convergence of (x_N, y_N) to (x_∞, y_∞) on $[0,1/2]$ (recall that $1/2$ is the time after which infinite clusters do appear) follows directly from the following lemma:

Lemma 37. *Uniformly on $[0,1/2]$, $\widehat{\gamma}_N$ converge to $\widehat{\gamma}_\infty$.*

This lemma is a direct consequence of the fact that $\mathcal{G}_t \subseteq \{\tau_v \leq t\}$, combination with the fact that for site Bernoulli percolation on $\widehat{\mathbb{T}}$ the probability that the size of the open cluster containing v_0 at time $1/2$ has volume bigger than N converges to 0 as $N \to \infty$.

It therefore remains to establish the convergence of x_N and y_N to x_∞ and y_∞ on the time interval $[1/2, 1]$.

Let us recall some useful facts from previous sections. Recall for $i, j \geq 0$ the numbers

$$a_{i,j} = |\{\mathbf{T} \text{ subtrees of } \widehat{\mathbb{T}} \text{ containing } v_0 : I(\mathbf{T}) = i, J(\mathbf{T}) = j\}| \tag{5.11}$$

and the bivariate generating function associated to them,

$$F(x,y) = \sum_{i,j\geq 0} a_{i,j} x^i y^j. \tag{5.12}$$

Recall that F has region of convergence

$$R = \{(x,y) \in \mathbb{R}^2 : \left| \frac{x}{(1-2y)^2} \right| \leq 1/4\}.$$

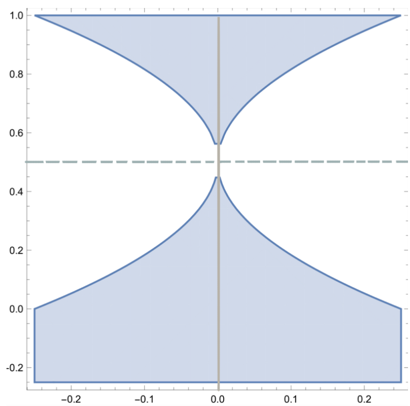

Figure 5.1: In blue the region of convergence of F

We define the partial sums for all $x, y \geq 0$

$$F_N(x,y) := \sum_{\substack{i,j\geq 0 \\ 2i+j+1<N}} a_{i,j} x^i y^j. \tag{5.13}$$

A first step in the proof is to express the function $\widehat{\gamma}_N$ in terms of (x_N, y_N) using the generating function F_N:

Lemma 38. *For all $t \in [0,1]$, the functions $\widehat{\gamma}_N$ satisfies the equation*

$$\widehat{\gamma}_N(t) = t - x_N F_N(t x_N, y_N). \tag{5.14}$$

Proof. If $\mathcal{C}_t$ is non empty, we proved in Chapter 1 (see Eq. (1.37)) that

$$|\,\mathcal{C}_t\,| = 2I(\mathcal{C}_t) + J(\mathcal{C}_t) + 1 \tag{5.15}$$

for all $0 \le t \le 1$, where we recall that $I(\mathcal{C}_t)$ denotes the number of internal vertices of $\mathcal{C}_t$ and $J(\mathcal{C}_t)$ denotes the number of boundary vertices with exactly one neighboring vertex outside $\mathcal{C}_t$. The vertex v_0 is frozen (red) if its clock rang before time t and if the cluster $\mathcal{C}_t$ of v_0 does not have a size strictly smaller than N. This implies that

$$\widehat{\gamma}_N(t) = t - \mathbf{P}_N(v_0 \in \mathcal{C}_t,\, 2I(\mathcal{C}_t) + J(\mathcal{C}_t) + 1 < N) \tag{5.16}$$

for all $t \in [0,1]$. By Lemma 34, with a_{ij} as in (5.11), the RHS above can be rewritten as

$$t - x_N \sum_{\substack{i,j \ge 0 \\ 2i+j+1<N}} a_{i,j} (2y_N)^j (t x_N)^i, \tag{5.17}$$

and we finally obtain (5.14) for $t \in [0,1]$. $\square$

The following upper bound for y_N will be useful:

Lemma 39. *There exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$, and for all $t \in [0,1]$,*

$$y_N(t) \le 7/16. \tag{5.18}$$

Proof. From Lemma 37, by dominated convergence since $0 \le \widehat{\gamma}_N \le 1$ it also holds that

$$(x_N, y_N) \to (x_\infty, y_\infty) \tag{5.19}$$

on $[0, 1/2]$ as $N \to \infty$. Moreover, it is clear that $\widehat{\gamma}_N(t) \leq t$ for all $t \in [0, 1]$ (see (5.14)). Using this fact and the expression for y_N in Lemma 34 we obtain

$$y_N(t) \leq t(1-t) + \int_0^{1/2} \widehat{\gamma}_N(s) ds + \frac{t^2}{2} - \frac{1}{8} \leq \frac{3}{8} + o(1), \tag{5.20}$$

for all $t \in [0, 1]$. Then there exists $N_0 \in \mathbb{N}$ such that the right side on the above inequality is bounded from $7/16$. □

For $N \geq N_0$, this upper bound on y_N ensures that

$$r_N(t) := \frac{t x_N}{(1 - 2y_N)^2} \text{ with } t \in [0, 1] \tag{5.21}$$

is well-defined and finite.

From now, we will work with the pair (r_N, y_N) rather than (x_N, y_N). In order to rewrite $\widehat{\gamma}_N$ in terms of (r_N, y_N), we use the expression

$$a_{i,j} = c_i \binom{2i+j}{j} 2^j \tag{5.22}$$

obtained in (1.69). By Lemma 38, we have

$$\begin{aligned}
\widehat{\gamma}_N(t) &= t - x_N \sum_{\substack{i,j \geq 0 \\ 2i+j+1<N}} a_{i,j} (t x_N)^i y_N^j \\
&= t - \frac{x_N}{1 - 2y_N} \sum_{i \geq 0} c_i \frac{(t x_N)^i}{(1 - 2y_N)^{2i}} p_{i,N,y_N} \\
&= t - \frac{1}{t}(1 - 2y_N) r_N \sum_{i \geq 0} c_i \, p_{i,N,y_N} \, r_N^i
\end{aligned} \tag{5.23}$$

where we defined

$$p_{i,N,y} := \sum_{j \geq 0} \binom{2i+j}{j} (2y)^j (1 - 2y)^{2i+1} \mathbf{1}_{2i+j+1<N}. \tag{5.24}$$

for every $y < 1/2$. As already seen in the previous chapters, these coefficients have a probabilistic interpretation. Let $\mathbf{NegBin}(2i+1, 2y)$ be a negative binomial random variable with parameters $(2i+1, 2y)$, then

$$p_{i,N,y} = \mathbb{P}[2i + \mathbf{NegBin}(2i+1, 2y) < N].$$

For fixed $y < 1/2$ and N large, one can think of $i \mapsto p_{i,N,y}$ as a truncation function, behaving like the indicator function of $\{i \leq N/(1-2y)\}$. We introduce the function

$$C_{N,y}(r) := \sum_{i\geq 0} c_i p_{i,N,y} r^i \tag{5.25}$$

for $y < 1/2$ and $0 \leq r \leq 1$ and think of it as a truncated version of the generating function $C(x) = \sum_{i\geq 0} c_i x^i$ of the Catalan numbers. Using this notation, we can rewrite Equation (5.23) as

$$\widehat{\gamma}_N(t) = t - \frac{1}{t}(1-2y_N)\, r_N\, C_{N,y_N}(r_N). \tag{5.26}$$

Using this expression, we can obtain the following differential equations satisfied by (r_N, y_N).

Lemma 40. *On the interval $[0,1]$ the functions r_N, y_N are differentiable and satisfy*

$$r_N' = G(r_N, y_N, t) \quad \textit{and} \quad y_N' = H(r_N, y_N, t),$$

where for every $(r,y,t) \in [0,1] \times [0,1/2) \times (0,1]$

$$G(r,y,t) := \frac{r^2}{t}(2 - C_{N,y}(r))^2 + \frac{2r}{1-2y}(2 - C_{N,y}(r))$$
$$+ (\frac{r}{t} + \frac{t}{1-2y})(1-4r),$$
$$H(r,y,t) := 1 - t - \frac{1}{t}\,(1-2y)\, rC_{N,y}(r).$$

Proof. Recall that

$$x_N(t) = \int_0^t (1 - t + \widehat{\gamma}_N(s))^2 ds \tag{5.27}$$

and

$$y_N(t) = \int_0^t (1 - t + \widehat{\gamma}_N(s)) ds. \tag{5.28}$$

The differentiability of the function $r_N(t) = tx_N/(1-2y_N)^2$ follows from the fact that y_N is bounded away from $1/2$ and from the differentiability of the functions in (5.27) and (5.28) since $\widehat{\gamma}_N \in [0,1]$ and it is continuous.

Differentiating (5.28) with respect to t, and plugging in the expression (5.26) for $\widehat{\gamma}_N$, we get

$$y_N' = 1 - t - \frac{1}{t} r_N (1 - 2y_N) C_{N,y_N}(r_N). \tag{5.29}$$

Differentating (5.27) with respect to t, we get

$$x_N'(t) = (1 - t + \widehat{\gamma}_N(t))^2 - 2\int_0^t (1 - t + \widehat{\gamma}_N(s)) ds, \tag{5.30}$$

which can be rewritten in terms of $y_N(t)$ and $y_N'(t)$ as

$$x_N'(t) = (y_N'(t) + t)^2 - 2y_N(t). \tag{5.31}$$

This yields to the expression for $(tx_N(t))'$,

$$(tx_N(t))' = t((y_N'(t) + t)^2 - 2y_N(t)) + x_N. \tag{5.32}$$

By using

$$r_N'(t) = \left(\frac{(tx_N(t))'}{x_N(t)} - \frac{4y_N'(t)}{(2y_N(t) - 1)} \right) r_N(t), \tag{5.33}$$

and equations (5.29) and (5.32), after some manipulations we obtain that

$$\begin{aligned}
&t(1-2y_N)^2 r'_N(t)\\
&= (2-C_{N,y_N}(r_N))\left[(2-C_{N,y_N}(r_N))((1-2y_N)r_N)^2+2t(1-2y_N)r_N\right]\\
&+(1-4r_N)(r_N(1-2y_N)^2+t^2(1-2y_N)),
\end{aligned}$$

which concludes the proof. □

5.4 Conclusion of the proof

We are now finally in a position to prove Lemma 36.

Conclusion of the proof of Lemma 36. Recall that it only remains to prove the uniform convergence on the interval $[1/2, 1]$. To achieve this, it suffices to prove that (r_N, y_N) converges uniformly to $(1/4, y_\infty)$ on this interval. First we will prove that r_N converges to $1/4$ using the first differential equation in Lemma 40 and applying the "fence" method, just as in the previous chapter. Then, we will show that y_N converges to y_∞ using the second differential inequality.

For every N and $y \in [0, 1/2)$ The function $C_{N,y}$ is continuous increasing and satisfies $C_{N,y}(1/4) < C(1/4) = 2$ and $C_{N,y}(1) > 2$ provided N large enough. Hence, for every N large and $t \in [0, 1]$, we can define $w_N(t) \in (1/4, 1]$ such that

$$C_{N,y_N(t)}(w_N(t)) = 2. \tag{5.34}$$

Observe that (w_N) converges uniformly to the constant function $1/4$ on $[0, 1]$. To see this, define $1/4 < \overline{w}_N \leq 1$ by $C_{N,7/16}(\overline{w}_N) = 2$. One can directly check from the definition of $C_{N,7/16}$ that

$$\lim_{N\to\infty} \overline{w}_N = \frac{1}{4}. \tag{5.35}$$

Furthermore, for N large enough, Lemma 39 gives $y_N \leq 7/6$ uniformly. Hence, for every t, $C_{N,y_N(t)}(\overline{w}_N) \geq C_{N,7/6}(\overline{w}_N) = 2$ which implies that for every $t \in [0,1]$

$$\frac{1}{4} \leq w_N(t) \leq \overline{w}_N. \tag{5.36}$$

Therefore, the uniform convergence of r_N to $r_\infty = 1/4$ on $[1/2,1]$ follows from the bounds

$$\forall t \in [1/2,1] \quad \frac{1}{4} \leq r_N(t) \leq w_N(t), \tag{5.37}$$

which will be established using the "fence" method as in the previous chapter. We begin with the lower bound. For every $0 < t \leq 1/2$, we have $\gamma_N(t) > 0 = \gamma_\infty(t)$. Hence, it follows from the expressions of x_N, y_N in Lemma 34 and x_∞, y_∞ in (3.29) that for every $0 < t \leq 1/2$, $x_N(t) > x_\infty(t)$ and $y_N(t) > y_\infty(t)$. In particular, defining $a_N := r_N(1/2)$, we have

$$a_N = \frac{x_N}{2(1-2y_N)^2} > \frac{x_\infty}{2(1-2y_\infty)^2} = r_\infty(1/2) \overset{(5.6)}{=} 1/4.$$

On the interval $[1/2,1]$, the function r_N is the unique solution of the ODE

$$\begin{cases} r' = G(r, y_N(t), t) & t \in [1/2,1] \\ r(1/2) = a_N. \end{cases} \tag{5.38}$$

One can easily check from the definition of G that for every $t \in [1/2,1]$, $G(1/4, y_N(t), t) > 0$ and we have seen that $a_N > 1/4$. Therefore, the constant function $r = 1/4$ is a lower fence for r_N on $[1/2,1]$, which establishes the lower bound in (5.37). For the upper bound, we wish to prove that $w_N(t)$ is an upper fence for r_N on the interval $[1/2,1]$. If we try to mimic the proof above, we would need to show that $a_N \leq w_N(1/2)$, which does not follow directly from the definitions. In order to circumvent this difficulty, we work on a larger interval for t and prove that the bound $r_N \leq w_N$ is true on $[0,1]$. More precisely, on $[0,1]$, the function r_N is the unique solution of the ODE

$$\begin{cases} r' = G(r, y_N(t), t) & t \in [0,1] \\ r(0) = 0. \end{cases} \tag{5.39}$$

Using $w_N(t) > 1/4$, one can easily check from the definition of G that for every $t \in [0,1]$, $G(w_N(t), y_N(t), t) < 0$. Furthermore, we also have $w_N(0) > 1/4 > 0$, which concludes that w_N is an upper fence for r_N on $[0,1]$.

It remains to prove the convergence of y_N to y_∞. From (5.2) and (3.29), we have the explicit formula

$$\forall t \in [1/2, 1] \quad y_\infty(t) = \frac{1}{2} - t^2 + t\log(2t). \tag{5.40}$$

Therefore, we can check that on the interval $[1/2, 1]$, the function y_∞ satisfies the differential inequality

$$y'_\infty = 1 - t - \frac{1}{2t}(1 - 2y_\infty). \tag{5.41}$$

Notice that this can be seen as the "limit" of the equation satisfied by y_N in Lemma 40. Indeed, since $r_\infty = 1/4$ and $C(1/4) = 2$, the equation above can be rewritten as

$$y'_\infty = 1 - t - \frac{1}{t} r_\infty (1 - 2y_\infty) C(r_\infty). \tag{5.42}$$

Define for $t \geq 1/2$,

$$\varepsilon_N(t) := y_N(t) - y_\infty(t). \tag{5.43}$$

Using Lemma 40 and Equation (5.41) above, we have for every $t \geq 1/2$

$$\frac{d}{dt}\left(\frac{\varepsilon(t)}{t}\right) = \alpha_N(t), \tag{5.44}$$

where

$$\alpha_N := \frac{1}{2t^2}(1 - 2r_N C_{N,y_N}(r_N))(1 - 2y_N).$$

Integrating this equation above $1/2$ and $t \geq 1/2$, we obtain

$$\varepsilon_N(t) = 2t\varepsilon_N(1/2) + t\int_{1/2}^{t} \alpha_N(s)ds. \tag{5.45}$$

By Lemma 37, $\varepsilon_N(1/2)$ converges to 0 as N tends to infinity. Hence, in order to conclude the proof it suffices to prove that α_N converges uniformly to 0 on $[1/2, 1]$. The bounds in (5.37) and (5.36) imply that for N large enough and every $t \geq 1/2$,

$$1 - 4\overline{w}_N \leq 1 - 2r_N(t)C_{N,y_N(t)}(r_N(t)) \leq 1 - \frac{1}{2}C_{N,7/16}(1/4). \tag{5.46}$$

Since $\overline{w}_N$ converges to $1/4$ and $C_{N,7/16}(1/4)$ converges to 2 as N tends to infinity, the sequence $(1 - 2r_N C_{N,y_N}(r_N))$ converges uniformly to 0 on $[1/2, 1]$. The sequence of functions α_N therefore also converges uniformly to 0 on $[1/2, 1]$, which in turn completes the proof of the lemma. $\square$

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